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ON ORBITAL TOPOLOGIES

By MARY POWDERLY (*Columbia University*)
and HING TONG (*Wesleyan University*)

[Received 27 April 1955; in revised form 17 September 1955]

LET γ be a mapping of a set S into S. Recently D. O. Ellis† raised the question of whether or not there exists a topology over S such that the topology is in general non-trivial and renders γ continuous. Our answer is in the affirmative.

Let S be any abstract set, γ any mapping of S into S. We shall now introduce a topology $S(\gamma)$ on S. Let $p = \gamma^n(x)$, where n is the smallest non-negative integer satisfying this equation. We shall then say that n is the 'order of (p, x) with respect to γ' or simply the 'order of (p, x)' when no possibility of confusion arises. Let $A \subset S$. We define

$A' = \{p \in S \mid \exists$ infinitely many $x_i \in A$ $(i = 1, 2, 3,...) \ni$ the orders of $(p, x_i) = n_i$ form a strictly increasing sequence of non-negative integers$\}$.

Let $\bar{A} = A + A'$. It is readily verified that (1) $\overline{A+B} = \bar{A}+\bar{B}$, (2) $A \subset \bar{A}$, (3) $\bar{\phi} = \phi$. We next show that $\bar{\bar{A}} \subset \bar{A}$. If this is done, then we have a topology which we designate $S(\gamma)$. Let $p \in \bar{\bar{A}}$. Hence there exist infinitely many $x_i \in \bar{A}$ $(i = 1, 2, 3,...)$ such that the orders of $(p, x_i) = n_i$ form a strictly increasing sequence of non-negative integers. Since $x_i \in \bar{A}$, either there is $x_{i_0} \in A'$ or no $x_i \in A'$. In the second case, obviously, $p \in \bar{A}$. In the first case, there are infinitely many $y_{i_0 j}$ $(j = 1, 2, 3,...) \ni y_{i_0 j} \in A$, and there is a strictly increasing sequence

$$n_{i_0 j} \ni x_{i_0} = \gamma^{n_{i_0 j}}(y_{i_0 j}).$$

Since $p = \gamma^{n_{i_0}}(x_{i_0})$, then

$$p = \gamma^{n_{i_0}+n_{i_0 j}}(y_{i_0 j}).$$

Let the order of $(p, y_{i_0 j})$ be $m_{i_0 j}$. Either there are an infinite number of j such that $m_{i_0 j}$ is strictly increasing or there is a non-negative α such that $m_{i_0 j} \leqslant \alpha$ for all j. In the first case $p \in \bar{A}$, according to the definition of $\bar{A}$. In the second case, there is an integer $\beta \ni m_{i_0 j} = \beta$ for infinitely many j; thus, $p = \gamma^{\beta}(y_{i_0 j})$ for infinitely many j. Since $\{n_{i_0 j}\}$ is a strictly increasing sequence, there is an $n_{i_0 j_0}$ such that $n_{i_0 j_0} > \beta$ and $p = \gamma^{\beta}(y_{i_0 j_0})$. Now

$$x_{i_0} = \gamma^{n_{i_0 j_0}}(y_{i_0 j_0}) = \gamma^{(n_{i_0 j_0}-\beta)+\beta}(y_{i_0 j_0}).$$

But

$$p = \gamma^{\beta}(y_{i_0 j_0}) = \gamma^{\beta}(y_{i_0 j})$$

† *Quart. J. of Math.* (Oxford) (2) 4 (1953), 117–19.

for infinitely many j. Hence we have $x_{i_0} = \gamma^{n_{i_0 j_0}}(y_{i_0 j})$ for infinitely many j. This contradicts the fact that the order of $(x_{i_0}, y_{i_0 j})$ is a strictly increasing sequence.

THEOREM 1. *$S(\gamma)$ is a T_1 space, but in general not a T_2.*

Proof. Let $x \in S(\gamma)$. Then $\{\bar{x}\} = \{x\}$, as is evident from the definition of $\bar{A}$. The second part of the statement can be readily seen by constructing an example. The construction is left to the reader.

THEOREM 2. *γ is continuous over $S(\gamma)$.*

Proof. We shall show that, for every $A \subset S$, we have $\gamma\bar{A} \subset \overline{\gamma A}$. Let $p \in \gamma\bar{A} - \gamma A$. Thus

$$\exists\, q \in \bar{A} - A \ni p = \gamma q.$$

Thus $q \in A'$. According to the definition of A', there are infinitely many $x_i \in A$ $(i = 1, 2, 3, \ldots)$ and a strictly increasing sequence of non-negative integers $n_i \ni q = \gamma^{n_i}(x_i)$, where the order of (q, x_i) is n_i. We show next that $p \in \overline{\gamma A}$. This will suffice to complete the proof of the theorem. Now,

$$p = \gamma(q) = \gamma(\gamma^{n_i}(x_i)) = \gamma^{n_i}(\gamma(x_i)).$$

If the orders of $(p, \gamma(x_1))$, $(p, \gamma(x_2))$,... have no upper bound, then $p \in \overline{\gamma A}$. We assume the contrary; then there is a positive integer η such that $k_i \leqslant \eta$ for all i, where k_i is the order of $(p, \gamma(x_i))$. The last fact implies that there is a non-negative integer $\kappa \ni k_i = \kappa$ for infinitely many i. Thus, $p = \gamma^{\kappa}(\gamma(x_i))$ or $p = \gamma^{\kappa+1}(x_i)$ for infinitely many i. Since n_i is a strictly increasing sequence,

$$\exists\, n_{i_0} \ni n_{i_0} > \kappa + 1$$

and $p = \gamma^{\kappa+1}(x_{i_0})$. Now,

$$\begin{aligned} q = \gamma^{n_{i_0}}(x_{i_0}) &= \gamma^{n_{i_0}-(\kappa+1)+\kappa+1}(x_{i_0}) \\ &= \gamma^{n_{i_0}-(\kappa+1)}\gamma^{\kappa+1}(x_{i_0}) = \gamma^{n_{i_0}-(\kappa+1)}\gamma^{\kappa+1}(x_i), \end{aligned}$$

for infinitely many i: that is, $q = \gamma^{n_{i_0}}(x_i)$ for infinitely many i. The last fact contradicts the strict monotonicity of the n_i.

In a subsequent note we hope to present further details concerning $S(\gamma)$.

SOLVABILITY OF CERTAIN EQUATIONS IN A FINITE FIELD

By L. CARLITZ (*Duke University*)

[Received 2 September 1955]

1. LET $q = p^n$, where p is a prime, and let $GF(q)$ denote the finite field of order q. Schwarz (1) has given a very elegant proof of the following theorem. *If $k \mid p-1$, if $a_1,..., a_k$ are non-zero numbers of $GF(q)$, and a is arbitrary, then the equation*

$$a_1 x_1^k+...+a_k x_k^k = a \tag{1}$$

has at least one solution in the field.

I wish to point out that, by the same method, the following generalization can be proved.

THEOREM 1. *Let $k \mid p-1$ and let $a_1,..., a_k$ be non-zero numbers of $GF(q)$. Let $g(x_1,..., x_k)$ be an arbitrary polynomial with coefficients in $GF(q)$ of degree less than k. Then the equation*

$$a_1 x_1^k+...+a_k x_k^k = g(x_1,..., x_k) \tag{2}$$

has at least one solution in the field.

Proof. If the theorem is false,

$$\{a_1 x_1^k+...+a_k x_k^k-g(x_1,..., x_k)\}^{q-1} = 1$$

for all $x_1,..., x_k$ in $GF(q)$. Consequently the sum

$$\sigma = \sum_{x_1,..., x_k} \{a_1 x_1^k+...+a_k x_k^k-g(x_1,..., x_k)\}^{q-1} = 0, \tag{3}$$

the sum extending over all $x_i \in GF(q)$. On the other hand we may expand by the multinomial theorem before summing. We recall that for arbitrary $m \geqslant 1$

$$\sum_{x \in GF(q)} x^m = \begin{cases} -1 & (q-1 \mid m), \\ 0 & \text{(otherwise)}. \end{cases} \tag{4}$$

Thus, in the expansion of the sum in (3), the only non-vanishing term is

$$\frac{(q-1)!}{\{((q-1)/k)!\}^k} (a_1...a_k)^{(q-1)/k}(-1)^k. \tag{5}$$

But the hypothesis $k \mid p-1$ implies that the multinomial coefficient in (5) is not divisible by p, so that $\sigma \neq 0$. This plainly contradicts (3).

As an immediate corollary of the theorem we may state the following. *If $f_1(x_1),..., f_k(x_k)$ are arbitrary polynomials with coefficients in $GF(q)$ and each of degree k, then the equation*

$$f_1(x_1)+...+f_k(x_k) = 0$$

has at least one solution in the field.

2. Theorem 1 can be extended without much difficulty. Consider the equation

$$f(x_1,\dots,x_k) = g(x_1,\dots,x_k), \tag{6}$$

where f is homogeneous of degree k, while g is arbitrary of degree less than k. With exactly the same method of proof, it evidently suffices to isolate the term $x_1^{q-1}\dots x_k^{q-1}$ in the expansion of $(f-g)^{q-1}$, or, what is the same thing, in the expansion of f^{q-1}. Again, applying (4), we see that this coefficient is

$$(-1)^k \sum_{x_1,\dots,x_k} f^{q-1}(x_1,\dots,x_k).$$

We may accordingly state

THEOREM 2. *If $f(x_1,\dots,x_k)$ is homogeneous of degree k while $g(x_1,\dots,x_k)$ is arbitrary of degree less than k, and*

$$\sum_{x_1,\dots,x_k} f^{q-1}(x_1,\dots,x_k) \neq 0, \tag{7}$$

then the equation (6) *has at least one solution in the field. Alternatively* (7) *may be replaced by the equivalent condition that the number of solutions of $f(x_1,\dots,x_k) = 0$ is not divisible by p.*

As a simple special case of Theorem 2 it is clear that the equation

$$x_1 x_2 \dots x_k = g(x_1,\dots,x_k) \quad (\deg g(x_1,\dots,x_k) < k) \tag{8}$$

has at least one solution. A more elaborate example including both (2) and (8) is furnished by the equation

$$a_1 x_1^k + \dots + a_k x_k^k + a x_1 x_2 \dots x_k = g(x_1,\dots,x_k). \tag{9}$$

The condition (7) now becomes

$$\sum_r \frac{(q-1)!}{(r!)^k (q-1-rk)!} (a_1 \dots a_k)^r a^{q-1-rk} \neq 0. \tag{10}$$

In particular (10) is satisfied when $a \neq 0$ but at least one $a_i = 0$.

As an example of the alternative condition in Theorem 2 we may take an irreducible factorable polynomial

$$f(x_1,\dots,x_k) = \prod_{j=0}^{k-1} (w_1^{q^j} x_1 + \dots + w_k^{q^j} x_k), \tag{11}$$

where $w_1,\dots, w_k$ are numbers of $GF(q^k)$ that are linearly independent relative to $GF(q)$. The polynomial (11) vanishes only at $(0,\dots,0)$.

REFERENCE

1. St. Schwarz, 'On the equation $a_1 x_1^k + a_2 x_2^k + \dots + a_k x_k^k + b = 0$ in finite fields', *Quart. J. of Math.* (Oxford), 19 (1948), 160–3.

ON MINIMAL SETS IN DYNAMICAL SYSTEMS

By YAEL NAIM DOWKER (*London*)

[Received 7 September 1955]

1. Introduction

A COMPACT discrete dynamical system (X, T) is a compact metric space X with a homeomorphism T of X onto itself. A subset A of X is called *invariant* (under T) if $TA = A$. A closed invariant subset M of X which contains no proper closed invariant subset is called a *minimal set* [(**1**), 198].

For each point $p \in X$, the set

$$O_p = \bigcup_{n=-\infty}^{\infty} T^n p$$

is called the *orbit* of p and the sets

$$O_p^+ = \bigcup_{n=0}^{\infty} T^n p, \qquad O_p^- = \bigcup_{n=0}^{-\infty} T^n p$$

are called respectively the *positive semi-orbit* of p and the *negative semi-orbit* of p. The set ω_p consisting of all limit points of the sequence $p, Tp, T^2p,...$ is called the *ω-limit set* of p and the set α_p of limit points of the sequence $p, T^{-1}p, T^{-2}p,...$ is called the *α-limit set* of p [(**1**), 198].

It is easy to see [(**1**), 198] that M is minimal if and only if, for every $p \in M$,

$$M = \bar{O}_p = \bar{O}_p^+ = \bar{O}_p^- = \omega_p = \alpha_p.$$

It is clear from the definition that, if M is a minimal set and A is any closed invariant set in X, then either $M \cap A = \varnothing$ or $M \cap A = M$. It is known that, for every point $p \in X$, ω_p contains at least one minimal set [(**1**), 200].†

A point $p \in X$ is called *positively semi-asymptotic* to a minimal set M if M is the only minimal set contained in ω_p and $p \notin M$ [(**1**), 204]. A minimal set M will be called a *tailed* minimal set if there exists a point $p \in X$ which is positively semi-asymptotic to M.

The space X is said to be *T-connected* if, for every closed proper subset A of X, $TA \cap \overline{CA} \neq \varnothing$, where CA denotes the complement of A in X [(**3**), 169].

In this paper a metric is defined in the space S of closed invariant sets of X so that S becomes a complete metric space. With this metric the space Ω of minimal sets of X is closed in S and hence is also a complete space. This metric induces the same topology in Ω as that naturally

† $\bar{A}$ denotes the closure of the set A. $\varnothing$ denotes the empty set.

induced by the neighbourhoods of minimal sets in X. It is also the same topology as that induced in Ω by the Hausdorff metric in the space of closed sets of X.

Baire's theorem applied to the complete space Ω leads to the following results. Assume that X is not itself a minimal set and does not contain a non-countable infinity of minimal sets. If X is connected or if X contains no isolated minimal set or if some point p of X is contained in no minimal set, then X contains at least one tailed minimal set; and, if X is T-connected, then either there are finitely many minimal sets and all are tailed or there are infinitely many minimal sets and infinitely many are tailed.

Applications to the theory of invariant measures in dynamical systems will be given elsewhere.

2. On minimal sets

LEMMA 1. *If M is minimal and $\epsilon > 0$, then there exists an integer N such that $p, Tp,..., T^N p$ is ϵ-dense in M for every $p \in M$.*

Proof. Let $\epsilon > 0$ be given. For every point $p \in M$ we have $\overline{O_p^+} = M$. Hence for every p there exists an integer n_p such that $p, Tp,..., T^{n_p}p$ is $\frac{1}{2}\epsilon$-dense in M. Suppose now that there exists no such N as required in the lemma. Then for every N there exists a point p_N such that $p_N, Tp_N,..., T^N p_N$ is not ϵ-dense in M. Since X is compact, there exists a sequence of integers $N(i)$ such that the sequence of points $p_{N(i)}$ converges to a limit, say p. For arbitrary n let $N(i) > n$ and be such that

$$\rho(T^j p, T^j p_{N(i)}) < \tfrac{1}{2}\epsilon$$

for $j = 1, 2,..., n$.

Now, since
$$p_{N(i)}, Tp_{N(i)},..., T^n p_{N(i)},..., T^{N(i)} p_{N(i)},$$
is not ϵ-dense in M, there exists a point q such that

$$\rho(q, T^j p_{N(i)}) > \epsilon \quad \text{for } j = 0, 1, 2,..., n.$$

Then $$\rho(q, T^j p) \geqslant \rho(q, T^j p_{N(i)}) - \rho(T^j, T^j p_{N(i)}) \geqslant \epsilon - \tfrac{1}{2}\epsilon = \tfrac{1}{2}\epsilon$$
for $j = 1, 2,..., n$, i.e. $p, Tp,..., T^n p$ is not $\frac{1}{2}\epsilon$-dense in M, which is a contradiction. Thus the lemma is proved.

I introduce now the following sequence of pseudo-metrics in X. For every pair of points p and q in X and $n = 0, 1, 2,...$ define

$$\rho_n(p,q) = \max_{-n \leqslant i \leqslant n}\Big[\max\Big\{\min_{-n \leqslant j \leqslant n} \rho(T^i p, T^j q),\ \min_{-n \leqslant j \leqslant n} \rho(T^j p, T^i q)\Big\}\Big],$$

where ρ is the metric of X.

The following properties of ρ_n $(n = 0, 1, 2,...)$ can be easily verified:

(1) $\rho_n(p,q) = \rho_n(q,p) \quad (p, q \in X)$;

(2) $\rho_n(p,r) \leqslant \rho_n(p,q) + \rho_n(q,r) \quad (p, q, r \in X)$.

It can also be seen that $\rho_n(p,q)$ ($n = 0, 1, 2,\ldots$) is positive unless $p = q$ or possibly if p and q are in the same orbit of a periodic point. In fact, it can be seen that, if we identify the points of each periodic orbit of period $\leqslant n$ in X, we obtain a metric space X_n with metric ρ_n. The space X_n is a decomposition space of X_{n-1} (and of X), i.e. a set of X_n is open if and only if the corresponding set in $X_{n-1}(X)$ is open.

If A is a subset of X, I shall denote the nth diameter of A by

$$\operatorname{diam}_n A = \sup_{p,q\in A} \rho_n(p,q).$$

We now introduce a sequence of distance functions in X by putting

$$d_n(p,q) = \max_{0\leqslant i\leqslant n} \rho_i(p,q)$$

for every pair of points p and q in X and $n = 0, 1, 2,\ldots$. Then, if p, q, r are arbitrary points in X we have

(1) (i) $d_n(p,q) > 0$ if $p \neq q$, because $d_n(p,q) \geqslant \rho_0(p,q) = \rho(p,q)$,
(ii) $d_n(p,p) = 0$, because $\rho_i(p,p) = 0$ ($i = 0, 1,\ldots, n$);

(2) $d_n(p,q) = d_n(q,p)$, because $\rho_i(p,q) = \rho_i(q,p)$ ($i = 0, 1,\ldots, n$);

(3) $d_n(p,r) \leqslant d_n(p,q)+d_n(q,r)$ because, if k is such that $0 \leqslant k \leqslant n$ and $d_n(p,r) = \rho_k(p,r)$, then $\rho_k(p,r) \leqslant \rho_k(p,q)+\rho_k(q,r)$ with $\rho_k(p,q) \leqslant d_n(p,q)$ and $\rho_k(q,r) \leqslant d_n(q,r)$.

Thus d_n defines a metric in X.

LEMMA 2. *d_n and ρ ($= \rho_0 = d_0$) induce the same topology in X.*

Proof. For any pair of points p and q and every positive integer n we have $d_n(p,q) \geqslant \rho(p,q)$. Hence d_n induces a finer topology in X than ρ.

On the other hand, let $\epsilon > 0$ and let $\delta > 0$ be such that $\rho(T^i p, T^i q) < \epsilon$ ($i = 0, \pm 1, \pm 2,\ldots, \pm k$) if $\rho(p,q) < \delta$. Then

$$\begin{aligned}\rho_k(p,q) &= \max_{-k\leqslant i\leqslant k}\Big[\max\Big\{\min_{-k\leqslant j\leqslant k} \rho(T^i p, T^j q),\ \min_{-k\leqslant j\leqslant k} \rho(T^j p, T^i q)\Big\}\Big] \\ &\leqslant \max_{-k\leqslant i\leqslant k} [\rho(T^i p, T^i q)] < \epsilon\end{aligned}$$

for $k = 0, 1,\ldots, n$. Hence $d_n(p,q) < \epsilon$ if $\rho(p,q) < \delta$. Thus ρ defines a finer topology in X than d_n. This finishes the proof of Lemma 2.

It follows directly from the definition of d_n that $d_n(p,q)$ is a non-decreasing sequence of positive numbers for every pair of points p and q in X.

Let now $F(X)$ be the space of all non-empty closed subsets of X. Following Hausdorff [(4), 146] we define in $F(X)$ a sequence of metrics H_n induced by d_n by putting

$$H_n(A, B) = \max\Big[\sup_{p\in A} d_n(p, B), \sup_{q\in B} d_n(q, A)\Big]$$

for every two sets A and B in $F(X)$. It is known [(**4**), 150] that $F(X)$ is a compact metric space with metric H_n ($n = 0, 1, 2,\ldots$).

Let now H_∞ be defined by putting

$$H_\infty(A, B) = \sup_n H_n(A, B)$$

for every two sets A and B in $F(X)$. Then H_∞ has the following properties:

(1) (i) $H_\infty(A, B) > 0$ if $A \neq B$, (ii) $H_\infty(A, A) = 0$,
(iii) $H_\infty(A, B) < \infty$;
(2) $H_\infty(A, B) = H_\infty(B, A)$;
(3) $H_\infty(A, C) \leqslant H_\infty(A, B) + H_\infty(B, C)$

for every A, B, and C in $F(X)$.

(1) (i), (ii) and (2) are immediate consequences of the definition of H_∞ in terms of H_n.

Proof of (1) (iii). X is compact and hence has a finite 0-diameter D. It follows from the definitions of ρ_n, d_n, H_n that $\rho_n(p, q) \leqslant D$ for every p and q in X, and hence $\mathrm{diam}_n X \leqslant D$ for $n = 0, 1, 2,\ldots$. Now

$$H_n(A, B) \leqslant \max_{0 \leqslant k \leqslant n} (\mathrm{diam}_n X) \leqslant D$$

for $n = 0, 1, 2,\ldots$ and A and B in $F(X)$; and (1) (iii) follows directly.

Proof of (3). Let A, B, C be sets in $F(X)$ and let $\epsilon > 0$ be given. Then there exists an integer $N = N(A, C, \epsilon)$ such that

$$\begin{aligned} H_\infty(A, C) &\leqslant H_N(A, C) + \epsilon \\ &\leqslant H_N(A, B) + H_N(B, C) + \epsilon \leqslant H_\infty(A, B) + H_\infty(B, C) + \epsilon, \end{aligned}$$

and (3) follows directly since ϵ is arbitrary.

LEMMA 3. *H_n ($n = 0, 1, 2,\ldots$) and H ($= H_0$) induce equivalent topologies in $F(X)$.*

Proof. For every pair of sets A and B in $F(X)$ the following holds: $H_n(A, B) > H_0(A, B)$ ($n = 0, 1, 2,\ldots$). Hence H_n induces a finer topology in $F(X)$ than H_0. Since $(F(X), H_n)$ is compact, it follows that H_n and H_0 induce equivalent topologies in $F(X)$.

THEOREM 1. *A necessary and sufficient condition that a non-empty closed invariant set M be a minimal set is that* $\mathrm{diam}_n M \to 0$ *as* $n \to \infty$.

Proof. Suppose first that M is a closed and invariant set such that

$\operatorname{diam}_n M \to 0$ as $n \to \infty$. Then, for every pair of points p and q in M, we have $\rho_n(p,q) \to 0$ as $n \to \infty$, i.e.

$$\max_{-n\leqslant i\leqslant n}\left[\max\left\{\min_{-n\leqslant j\leqslant n}\rho(T^i p, T^j q),\ \min_{-n\leqslant j\leqslant n}\rho(T^j p, T^i q)\right\}\right] \to 0 \quad \text{as } n \to \infty.$$

Hence
$$\min_{-n\leqslant j\leqslant n}\rho(q, T^j p) \to 0 \quad \text{as } n \to \infty,$$
i.e. q is on the orbit closure O_p of p. Since q is arbitrary, it follows that $\bar{O}_p = M$. Since p is arbitrary, it follows that $\bar{O}_p = M$ for every $p \in M$, and, by § 1, M is minimal.

Suppose now that M is a minimal set. Let $\epsilon > 0$ be given. Then by Lemma 1 there exists an integer N such that for every $p \in M$ we have that $p, Tp,\ldots, T^N p$, and hence also $p, Tp,\ldots, T^n p$, is ϵ-dense in M for every $n \geqslant N$. Let now p and q be two arbitrary points in M and let $n \geqslant N$. Then

$$\rho_n(p,q) = \max_{-n\leqslant i\leqslant n}\left[\max\left\{\min_{-n\leqslant j\leqslant n}\rho(T^i p, T^j q),\ \min_{-n\leqslant j\leqslant n}\rho(T^i p, T^j q)\right\}\right].$$

Now, for every i,
$$\rho(T^i p, T^j q) < \epsilon$$
for some $0 \leqslant j \leqslant n$ because $q, Tq,\ldots, T^n q$ is ϵ-dense in M; and
$$\rho(T^k p, T^i q) < \epsilon$$
for some $0 \leqslant k \leqslant n$ because $p, Tp,\ldots, T^n p$ is ϵ-dense in M. It follows that $\rho_n(p,q) < \epsilon$. Hence $\operatorname{diam}_n M = \sup_{p,q\in M}[\rho_n(p,q)] < \epsilon$ if $n \geqslant N$. This proves the theorem.

Lemma 4. *Let S be the space of non-empty invariant closed sets in X. Then S is a closed subspace of $F(X)$ with respect to the metric H.*

Proof. Let A be a limit point of S in $F(X)$. Then there exists a sequence $\{A_n\}$ of closed invariant sets such that $H(A_n, A) \to 0$ as $n \to \infty$. I shall prove that $A \in S$, i.e. A is invariant. Indeed let $\epsilon > 0$ be given and let $\delta > 0$ be such that $\rho(Tp, Tq) < \epsilon$ for every pair of points p and q in X such that $\rho(p,q) < \delta$. Let now N be a positive integer such that $H(A_n, A) < \delta$ for all $n > N$. Hence for every $p \in A$ we have $\rho(p, A_n) < \delta$ for all $n > N$. For each $n > N$ there exists a point $q_n \in A_n$ such that $\rho(p, A_n) = \rho(p, q_n)$. Then $\rho(p, q_n) < \delta$ and hence $\rho(Tp, Tq_n) < \epsilon$ for all $n > N$. Now $\rho(Tp, A_n) \leqslant \rho(Tp, Tq_n)$ since $Tq_n \in A_n$. Hence
$$\rho(Tp, A_n) < \epsilon$$
for all $n > N$ and all $p \in A$. Now
$$\sup_{p\in A}\rho(Tp, A_n) = \sup_{q\in TA}\rho(q, A_n),$$
and hence
$$\sup_{q\in TA}\rho(q, A_n) < \epsilon$$
for all $n > N$.

On the other hand, for every point $p \in A_n$ with $n > N$, we have $\rho(T^{-1}p, A) < \delta$. Let $n > N$, and, for every $p \in A_n$, let $q \in A$ be such that

$$\rho(T^{-1}p, A) = \rho(T^{-1}p, q).$$

Then $\rho(T^{-1}p, q) < \delta$, and hence $\rho(p, Tq) < \epsilon$ for all $n > N$. Thus $\rho(p, TA) \leqslant \rho(p, Tq) < \epsilon$ for all $p \in A_n$. Hence

$$\max_{p \in A_n} \rho(p, TA) < \epsilon$$

for all $n > N$. Then $H(A_n, TA) < \epsilon$ for all $n > N$. Thus $A_n \to TA$ as $n \to \infty$. But $A_n \to A$ as $n \to \infty$, and hence $A = TA$, i.e. $A \in S$.

As a direct consequence of Lemmas 3 and 4 we obtain the following corollary.

COROLLARY 1. *S is a closed subset of $F(X)$ with respect to the metric H_n for each $n = 0, 1, 2, \ldots$.*

Since $F(X)$ is a compact metric space for each of the metrics H_n ($n = 0, 1, \ldots$), we obtain

COROLLARY 2. *S is a compact and hence complete metric space with respect to the metric H_n for $n = 0, 1, 2, \ldots$.*

LEMMA 5. *S is a complete metric space with respect to the metric H_∞.*

Proof. Let $\{A_n\}$ be a Cauchy sequence of sets in (S, H_∞), i.e. for every $\epsilon > 0$ there exists an integer N such that $H_\infty(A_n, A_m) < \epsilon$ for every pair of integers $n, m > N$. Then $H_k(A_n, A_m) < \epsilon$ for every pair of integers $n, m > N$ and $k = 0, 1, 2, \ldots$. Thus in particular $\{A_n\}$ is a Cauchy sequence in (S, H). Hence, by Corollary 2, there exists a set $A \in S$ such that $H(A_n, A) \to 0$ as $n \to \infty$. Hence, by Lemma 3, $H_k(A_n, A) \to 0$ as $n \to \infty$ for $k = 0, 1, 2, \ldots$.

Let now $\epsilon > 0$ be given and let K be a positive integer such that $H_\infty(A_n, A_m) < \frac{1}{2}\epsilon$ if $n, m > K$. Then $H_k(A_n, A_m) < \frac{1}{2}\epsilon$ for all $n, m > K$ and $k = 0, 1, 2, \ldots$. For every k let n_k be an integer such that $n_k > K$ and $H_k(A, A_{n_k}) < \frac{1}{2}\epsilon$. Then

$$H_k(A, A_n) \leqslant H_k(A, A_{n_k}) + H_k(A_{n_k}, A_n)$$

which is less than $\frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ if $n > K$. Hence $H_\infty(A, A_n) < \epsilon$ if $n > K$, i.e. A is the limit of A_n in (S, H_∞).

LEMMA 6. *Let Ω be the space of all minimal sets in X. Then (Ω, H_∞) is a closed subspace of (S, H_∞).*

Proof. Let A be a limit point of (Ω, H_∞) in (S, H_∞). Then there exists a sequence $\{M_n\}$ of minimal sets such that $H_\infty(M_n, A) \to 0$ as $n \to \infty$, i.e. for every $\epsilon > 0$ there exists an integer K such that $H_\infty(M_n, A) < \frac{1}{3}\epsilon$ for

all $n > K$. Hence $H_k(M_n, A) < \frac{1}{3}\epsilon$ if $n \geqslant K$ and $k = 0, 1, 2,\ldots$. By Lemma 1 there exists an integer N such that $q, Tq,\ldots, T^N q$ is $\frac{1}{3}\epsilon$-dense in M_K for every $q \in M_K$. Let p, p' be any two points in A. Then

$$\min_{-N\leqslant j\leqslant N} \rho(T^j p, p') \leqslant \min_{-N\leqslant j\leqslant N} \rho(T^j p, T^i q)+\rho(T^i q, q')+\rho(q', p')$$
$$\leqslant d_N(p,q)+\rho(T^i q, q')+d_N(p', q'),$$

where $q \in M_K$ and is such that $d_N(p, M_K) = d_N(p,q)$, $q' \in M_K$ and is such that $d_N(p', M_K) = d_N(p', q')$, and i is such that $0 \leqslant i \leqslant N$ and $\rho(T^i p, q') < \frac{1}{3}\epsilon$. Then $d_N(p,q) < \frac{1}{3}\epsilon$, $d_N(p', q') < \frac{1}{3}\epsilon$ and hence

$$\min_{-N\leqslant j\leqslant N} \rho(T^j p, p') < \epsilon.$$

Since p' is an arbitrary point of A, it follows that $p, T^{\pm 1}p,\ldots, T^{\pm N}p$ is ϵ-dense in A, and, since ϵ is arbitrary, it follows that $\bar{O}_p = A$. Since p is an arbitrary point of A, it follows that $\bar{O}_p = A$ for every $p \in A$, i.e. A is minimal.

As a direct consequence of Lemmas 5 and 6 we obtain

THEOREM 2. *(Ω, H_∞) is a complete metric space.*

LEMMA 7. *Let M be a minimal set and let $\epsilon > 0$, then there exists an integer $N = N(M, \epsilon)$ such that*

$$H_\infty(M, M') < H_N(M, M')+\epsilon$$

for every minimal set M' of (X, T).

Proof. Let N be such that $p, Tp,\ldots, T^N p$ is ϵ-dense in M for every $p \in M$ [cf. Lemma 1 above]. In order to prove the lemma it is sufficient to prove that $H_m(M, M') \leqslant H_N(M, M')+\epsilon$ for every M. M' and $m = 0, 1, 2,\ldots$. If $m \leqslant N$, the lemma follows directly from the definitions of H_m and H_N. Let then $m > N$. Let $p \in M$ and $q \in M'$. Then

$$\rho_n(p,q) \leqslant d_N(p,q) \quad \text{if } n \leqslant N.$$

For $n > N$ we have for every integer i an integer $k(i)$ such that

$$0 \leqslant k(i) \leqslant N \quad \text{and} \quad \rho(T^i p, T^{k(i)} p) < \epsilon.$$

Hence

$$\min_{-n\leqslant j\leqslant n} \rho(T^i p, T^j q) \leqslant \rho(T^i p, T^{k(i)} p)+ \min_{-n\leqslant j\leqslant n} \rho(T^{k(i)} p, T^j q)$$
$$< \epsilon+ \min_{-N\leqslant j\leqslant N} \rho(T^{k(i)} p, T^j q) < \epsilon+\rho_N(p,q) < \epsilon+d_N(p,q)$$

for $i = 0, 1,\ldots$.

On the other hand for every integer i let $p_i \in M$ be such that

$$d_N(T^i q, M) = d_N(p_i, T^i q).$$

Then

$$\min_{-n\leqslant j\leqslant n} \rho(T^j p, T^i q) \leqslant \min_{-n\leqslant j\leqslant n} \rho(T^j p, p_i)+\rho(p_i, T^i q)$$
$$\leqslant \min_{-N\leqslant j\leqslant N} \rho(T^j p, p_i)+\rho(p_i, T^i q) \leqslant \epsilon+d_N(p_i, T^i q)$$
$$= \epsilon+d_N(T^i q, M) < \epsilon+H_N(M, M').$$

Thus

$$\rho_n(p,q) \leqslant \max[\epsilon+d_N(p,q), \epsilon+H_N(M,M')] \quad (n = 0, 1,...),$$

and hence

$$d_m(p,q) \leqslant \max[\epsilon+d_N(p,q), \epsilon+H_N(M,M')] \quad \text{for } m = 0, 1,...$$

and $p \in M$, $q \in M'$.

Now, for every $p \in M$, let $q(p) \in M'$ be such that

$$d_N(p, q(p)) = d_N(p, M').$$

Then

$$d_m(p, M') \leqslant d_m(p, q(p)) \leqslant \max[\epsilon+d_N(p, M'), \epsilon+H_N(M,M')]$$
$$= \epsilon+H_N(M, M')$$

for every $p \in M$, and hence $H_m(M, M') \leqslant H_N(M, M')+\epsilon$.

LEMMA 8. *H_∞ and H induce equivalent topologies in Ω.*

Proof. Since $H_\infty(M,N) \geqslant H(M,N)$ for every pair of sets M and N in Ω, it follows that H_∞ induces a finer topology in Ω than H. On the other hand, let $M \in \Omega$ and $\epsilon > 0$. By Lemma 7 there exists an integer $N = N(M, \epsilon)$ such that $H_\infty(M, M') \leqslant H_N(M, M')+\frac{1}{2}\epsilon$ for every $M' \in \Omega$. By Lemma 3 there exists a $\delta = \delta(N, \epsilon) > 0$ such that $H_N(M, M') \leqslant \frac{1}{2}\epsilon$ for every set $M' \in \Omega$ such that $H(M, M') < \delta$. Thus, if M' is any minimal set such that $H(M, M') < \delta$, then $H_\infty(M, M') < \epsilon$, i.e. H induces a finer topology in Ω than H_∞. Hence H and H_∞ induce equivalent topologies in Ω.

COROLLARY 3. *(Ω, H) is a topologically complete metric space.*

Proof. By Theorem 2 (Ω, H_∞) is a complete metric space. By Lemma 8 H and H_∞ induce the same topology in Ω.

LEMMA 9. *Let M be a minimal set and let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that, if M' is any minimal set in the δ-neighbourhood of M in X, then M is in the ϵ-neighbourhood of M in (Ω, H).*

Proof. Let N be an integer such that for every $p \in M$, p, Tp,..., $T^N p$ is $\frac{1}{2}\epsilon$-dense in M [cf. Lemma 1 above]. Let δ, $0 < \delta < \epsilon$, be such that $\rho(T^i p, T^i q) < \frac{1}{2}\epsilon$ $(i = 0, \pm 1, \pm 2,..., \pm N)$ for every pair of points p, q in X such that $\rho(p,q) < \delta$. Let now $M' \in \Omega$ be in the δ-neighbourhood

of M. Then $\rho(M,q) < \delta$ for every $q \in M'$. Let $p \in M$. Then there exists a point $q' \in M'$ such that $\rho(p, M') = \rho(p,q')$. There exists a point $p_1 \in M$ such that $\rho(M,q') = \rho(p_1,q') < \delta$. Let j $(0 \leqslant j \leqslant N)$ be such that $\rho(p, T^j p_1) < \frac{1}{2}\epsilon$. Then

$$\rho(p, M') \leqslant \rho(p, T^j q') \leqslant \rho(p, T^j p_1)+\rho(T^j p_1, T^j q') < \tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon = \epsilon.$$

Thus $$H(M, M') = \max\Big[\sup_{q\in M'} \rho(M,q), \sup_{p\in M} \rho(p, M')\Big] < \epsilon.$$

This completes the proof of Lemma 9.

For each open set U of X, let $V_U = \{M\colon M \in \Omega,\ M \subset U\}$. We define a topology V in Ω by taking all such sets V_U as a basis for open sets. I shall denote the fact that Ω is considered with this topology by writing (Ω, V).

It follows from Lemma 9 and the definition of the topology V in Ω that V and H induce equivalent topologies in Ω. It then follows by applying Lemma 8 that

THEOREM 3. *The three topologies defined in Ω by V, H, H_∞ respectively are mutually equivalent.*

Remark. It can be seen by examples that (Ω, H) is not necessarily complete: e.g. let (ρ, θ) be polar coordinates in the plane and let X be the closed disk $\rho \leqslant 1$. Let T be defined by $T(\rho,\theta) = (\rho, \theta+2\pi\rho)$. Let α_n be a sequence of irrational numbers such that $0 < \alpha_n < 1$ $(n = 1, 2,\ldots)$ and $\alpha_n \to r$ as $n \to \infty$, where r is a rational number.

If M_n is the circle $\rho = \alpha_n$ and A is the circle $\rho = r$, then $M_n \in \Omega$ and $H(M_n, A) \to 0$ as $n \to \infty$, i.e. A is the limit of M_n in (S, H). Hence M_n forms a Cauchy sequence in (Ω, H). But A is the union of a non-countable number of periodic orbits, and hence $A \notin \Omega$. Hence $\{M_n\}$ has no limit in (Ω, H), and hence (Ω, H) is not complete.

3. Applications

The results of the previous paragraph have several applications: some of them will be treated below. First we shall need the following lemma:

LEMMA 10. *If X is T-connected and A is a closed invariant proper subset of X, then A is not isolated and, if U is any open set such that $A \subset U \subseteq \overline{U} \subset X$, then there exists a compact set K such that $A \subset K \subseteq \overline{U}$ and $TK \subseteq K$.*†

† The symbol $\subset$ denotes 'strictly contained in'.

Proof. To show that A is not isolated we assume to the contrary that A is isolated. Then $TA \wedge CA = A \wedge CA = \emptyset$, which is in contradiction with the assumption that X is T-connected. Thus A is not isolated.

Let now U be an open set such that $A \subset U \subseteq \bar{U} \subset X$. Since X is T-connected, there exists by Theorem 1 of (**3**) and its proof a compact discrete dynamical system (Y, S) such that X is imbedded in Y as a compact proper subset, S coincides with T on X, and $X = \omega_q$ for a point $q \in Y$, $q \notin X$. Moreover $Y = X \cup O_q \cup P$ where P is a point such that $P \neq q$, $P \notin X$, $P = \alpha_q$. Let W be an open set in Y such that $W \wedge X = U$, $\bar{W} \wedge X = \bar{U}$, and $P \notin W$. Then, by a theorem due to Kerekjarto [cf. (**5**), (**2**) 126], either (i) there exists an open set $W_1 \subset W$ such that $T^{-1}W_1 \subseteq W_1$, or (ii) there exists a compact set K such that $A \subset K \subseteq \bar{W}$, $TK \subseteq K$. I shall first exclude the possibility of occurrence of (i). Indeed, suppose that (i) occurs. Then, since $X = \omega_q$ and $\omega_q \supset X - \bar{U} = X - \bar{W} \neq \emptyset$, there must be a positive integer j such that $T^j q \notin W_1$. It follows that $T^{j+r} q \notin W_1$ for $r = 1, 2, \ldots$ and hence $U \not\subset \omega_q$, which is absurd since $U \subset X$. Thus (i) cannot occur.

Hence (ii) occurs. Let K be a compact as described in (ii). Then $q_1 \notin K$ for every point $q_1 \in O_q$. For, if $q_1 \in K$, then, since $TK \subseteq K \subseteq \bar{W}$, it follows that $T^j q_1 \in \bar{W}$ for $j = 1, 2, \ldots$ and $X - \bar{W} = X - \bar{U} \subseteq \omega_q$, which contradicts the fact that $X = \omega_q$. It follows that $K \subseteq X$, and hence $A \subset K \subset X \wedge \bar{W} = \bar{U}$. This completes the proof of the lemma.

LEMMA 11. *Let X be T-connected and let M be a minimal set in X such that $X \neq M$ and M is isolated in (Ω, H_∞). Then M is tailed.*

Proof. Suppose that M is isolated in (Ω, H_∞), $M \subset X$. Then, by Theorem 3, M is also isolated in (Ω, V), i.e. there exists a neighbourhood U of M in X such that $M \subset U \subseteq \bar{U} \subset X$ and such that $\bar{U}$ contains no minimal sets other than M. By Lemma 10 there exists a compact set K such that $M \subset K \subseteq \bar{U}$, $TK \subseteq K$. Let $p \in K$, $p \notin M$. Then $\omega_p \subseteq K \subseteq \bar{U}$. Hence ω_p contains no minimal set other than M. On the other hand ω_p must contain at least one minimal set [(**1**), 199] and hence $\omega_p \supseteq M$. Hence p is positively asymptotic to M and M is tailed.

THEOREM 4. *Let X be T-connected and let X have α minimal sets where $1 < \alpha \leqslant \aleph_0$. Then α of the minimal sets are tailed.*

Proof. First let α be finite. Let $M_1, \ldots, M_\alpha$ be the minimal sets in X. Then Ω consists of the finite number of points $M_1, \ldots, M_\alpha$. Thus each M_j $(1 \leqslant j \leqslant \alpha)$ is isolated in (Ω, H_∞), and hence by Lemma 10 each minimal set M_j $(1 \leqslant j \leqslant \alpha)$ is tailed.

Let now $\alpha = \aleph_0$. Let $M_1, M_2, \ldots$ be the minimal sets of X. Ω then consists of an enumerable number of points $M_1, M_2, \ldots$. By Theorem 2, (Ω, H_∞) is complete and hence by Baire's theorem at least one point in Ω, say M_i, is not nowhere dense and hence is isolated in (Ω, H_∞). It follows easily that an enumerable number of points, say $M_{i_1}, M_{i_2}, \ldots$ are isolated in (Ω, H_∞). By Lemma 11 each of the $\aleph_0$ minimal sets $M_{i_1}, M_{i_2}, \ldots$ is tailed.

LEMMA 12. *Let there exist in X a point p which does not belong to any minimal set. Then there are either a non-countable number of minimal sets in X, or there exists a tailed minimal set in X.*

Proof. Let $p \in X$, $p \notin M$ for every minimal set M in X. Let α be the power of the set of minimal sets of ω_p. I shall show that, if $\alpha \leqslant \aleph_0$, then X contains a tailed minimal set. Indeed, if $\alpha = 1$, then ω_p contains one and one only minimal set M. Then p is positively semi-asymptotic to M, and M is tailed. If $1 < \alpha \leqslant \aleph_0$, we can apply Theorem 4 substituting ω_p for X since ω_p is T-connected [cf. Theorem 1 of (**2**)] and obtain the desired result.

THEOREM 5. *Let X be such that no minimal set is both open and closed in X. Then either X contains non-countably many minimal sets or X contains a tailed minimal set.*

Proof. Suppose the power of Ω is α, where $\alpha \leqslant \aleph_0$. Then as in the proof of Theorem 4 it follows from the completeness of (Ω, H_∞) that there exists a point $M \in \Omega$ which is isolated in (Ω, H_∞) and hence, by Theorem 3, also in (Ω, V): that is, there exists a neighbourhood U of M such that $M \subseteq U \subseteq \overline{U}$, where U contains no minimal sets M' with $M' \neq M$. By assumption M is not both open and closed in X. Hence $M \subset U$. Moreover we can assume with no loss of generality that $\overline{U}$ contains no minimal set M' with $M' \neq M$. Now by a theorem of Kerekjarto's (**5**) either (1) there exists an open set W such that $M \subset W \subseteq U$, $W \subseteq TW$ or (2) there exists a compact set K such that $M \subset K \subseteq \overline{U}$, $TK \subseteq K$.

Suppose first that (2) occurs and let $p \in K$, $p \notin M$. Then, as in the proof of Lemma 11, p is positively asymptotic to M, and hence M is tailed.

Suppose now that (1) occurs. Then two cases can be distinguished. (i) $\overline{W} \supseteq TW$ and (ii) $\overline{W} \not\supseteq TW$. In case (i) $\overline{W} = T\overline{W}$ and thus this case reduces to (2) which has been considered above. In case (ii) let

$$p \in TW - \overline{W}.$$

Then $T^j p \notin TW - \overline{W}$ ($j = 2, 3...$). Thus $p \notin \omega_p$ and hence belongs to no minimal set. The required result then follows directly from Lemma 12.

Applying Theorem 5 to a connected space X we obtain

COROLLARY 4. *If X is connected, then X is either minimal, or contains a non-countable number of minimal sets, or contains a tailed minimal set.*

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ON THE GENERATORS OF NILPOTENT LINEAR ALGEBRAS

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1. Introduction

A SET of elements $x_1,..., x_m$ of a linear algebra $\mathbf{A}$ is said to *generate* $\mathbf{A}$ if every element of $\mathbf{A}$ is linearly dependent on products of $x_1,..., x_m$; the elements $x_1,..., x_m$ are then called *generators* of $\mathbf{A}$. Knebelman (3) has defined the *nullity* of $\mathbf{A}$ to be the minimum number of linearly independent generators, and the *genus* of $\mathbf{A}$ to be the difference between its dimension and its nullity. In this paper generators of certain types of nilpotent linear algebras are studied. It is shown that, for these algebras, the genus is equal to the dimension of the first derived algebra.

The *right powers* of $\mathbf{A}$ are denoted by $\mathbf{A}^i$ $(i = 1, 2,...)$. They are defined inductively by

$$\mathbf{A}^1 = \mathbf{A}, \qquad \mathbf{A}^{i+1} = \mathbf{A}^i\mathbf{A} \quad (i \geqslant 1),$$

so that $\mathbf{A}^{i+1}$ is the subspace of $\mathbf{A}^i$ spanned by the elements of the form xy, where x belongs to $\mathbf{A}^i$ and y belongs to $\mathbf{A}$. The *left powers* of $\mathbf{A}$ are denoted by $\mathbf{A}_i$ $(i = 1, 2,...)$. They are defined by

$$\mathbf{A}_1 = \mathbf{A}, \qquad \mathbf{A}_{i+1} = \mathbf{A}\mathbf{A}_i \quad (i \geqslant 1).$$

An algebra $\mathbf{A}$ is said to be *nilpotent* (1) if $\mathbf{A}^m = 0 = \mathbf{A}_m$ for some integer m.

If $\mathbf{A}^{j+1} = \mathbf{A}^j$, then $\mathbf{A}^k = \mathbf{A}^j$ $(k \geqslant j)$. Otherwise $\mathbf{A}^{j+1} < \mathbf{A}^j$: that is, $\mathbf{A}^{j+1}$ is a proper subspace of $\mathbf{A}^j$. Hence, if the dimension of $\mathbf{A}$ is n, there is an integer r, not exceeding $n+1$, such that

$$\mathbf{A}^1 > \mathbf{A}^2 > \mathbf{A}^3 > ... > \mathbf{A}^r,$$
$$\mathbf{A}^k = \mathbf{A}^r \quad (k \geqslant r).$$

Similarly, there is an integer s, not exceeding $n+1$, such that

$$\mathbf{A}_1 > \mathbf{A}_2 > \mathbf{A}_3 > ... > \mathbf{A}_s,$$
$$\mathbf{A}_k = \mathbf{A}_s \quad (k \geqslant s).$$

If $\mathbf{A}$ is nilpotent, then clearly $\mathbf{A}^r$ and $\mathbf{A}_s$ are both zero.

The successive *derived algebras* of $\mathbf{A}$ are denoted by $\mathbf{A}^{(i)}$ $(i = 0, 1,...)$. They are defined (1) by

$$\mathbf{A}^{(0)} = \mathbf{A}, \qquad \mathbf{A}^{(i+1)} = \mathbf{A}^{(i)}\mathbf{A}^{(i)} \quad (i \geqslant 0).$$

Quart. J. Math. Oxford (2), 7 (1956), 17-23.

From this and the previous definitions, we have

$$\mathbf{A}^{(1)} = \mathbf{A}^2 = \mathbf{A}_2.$$

2. Canonical bases

The algebras discussed in § 1 are closed with respect to multiplication and satisfy the distributive laws. Multiplication is otherwise unrestricted; in particular, the associative law $(xy)z = x(yz)$ does not necessarily hold. However, for the purposes of the present paper, we shall assume that the following condition (which is satisfied in a large class of linear algebras) holds:

(α) *The set of subspaces* $\mathbf{A}^1,\ldots, \mathbf{A}^r, \mathbf{A}_1,\ldots, \mathbf{A}_s$ *can be ordered by the relation* '$\geqslant$'.

Since $\mathbf{A}^{i+1} < \mathbf{A}^i$ $(i = 1,\ldots, r-1)$ and $\mathbf{A}_{i+1} < \mathbf{A}_i$ $(i = 1,\ldots, s-1)$, condition (α) is equivalent to the assumption that any two subspaces $\mathbf{A}^j$, $\mathbf{A}_k$ satisfy either $\mathbf{A}^j \leqslant \mathbf{A}_k$ or $\mathbf{A}_k \leqslant \mathbf{A}^j$.

Let $\mathbf{B}(1),\ldots, \mathbf{B}(t)$ denote the distinct subspaces in the set $\mathbf{A}^1,\ldots, \mathbf{A}^r$, $\mathbf{A}_1,\ldots, \mathbf{A}_s$ arranged in descending order, so that

$$\mathbf{B}(i+1) < \mathbf{B}(i) \quad (i = 1,\ldots, t-1).$$

Let the dimension of $\mathbf{B}(i)$ be n_i. Then $n_1 = n$, and $n_{i+1} < n_i$.

It is always possible to choose a basis $e_1,\ldots, e_n$ for $\mathbf{A}$ such that $e_1,\ldots, e_{n_i}$ $(i = 1,\ldots, t)$ is a basis for $\mathbf{B}(i)$. Such a basis will be called a *canonical basis* for $\mathbf{A}$.

LEMMA. *Let* $\mathbf{B}(i)$ $(i > 1)$ *be a subspace such that* $\mathbf{B}(i) > \mathbf{A}^r$ *and* $\mathbf{B}(i) > \mathbf{A}_s$. *If* $e_1,\ldots, e_n$ *is a canonical basis for* $\mathbf{A}$, *there are independent elements* f_u $(u = n_{i+1}+1,\ldots, n_i)$ *of* $\mathbf{A}$ *satisfying the conditions*:

(i) *each element* f_u *is a product of the elements* $e_{n_i+1},\ldots, e_n$;

(ii) *the elements*

$$e_1,\ldots, e_{n_{i+1}}, f_{n_{i+1}+1},\ldots, f_{n_i}, e_{n_i+1},\ldots, e_n$$

form a canonical basis for $\mathbf{A}$.

Proof. Suppose that $\mathbf{B}(i) = \mathbf{A}^p$. Since $\mathbf{B}(i) > \mathbf{A}^r$, $p < r$. Hence $\mathbf{A}^p > \mathbf{A}^{p+1}$, and so products of the form

$$e_j x \quad (j = 1,\ldots, n_i)$$

are in $\mathbf{B}(i+1)$. Since $\mathbf{B}(i) > \mathbf{A}_s$, products of the form

$$x e_k \quad (k = 1,\ldots, n_i)$$

are also in $\mathbf{B}(i+1)$.

For some integer $h < i$, $\mathbf{A}^{p-1} = \mathbf{B}(h)$. Then $\mathbf{B}(i)$, since it is the subspace $\mathbf{A}^p$, is spanned by

$$e_j e_k \quad (j = 1,\ldots, n_h;\ k = 1,\ldots, n).$$

But, as we have seen, products of this form are in $\mathbf{B}(i+1)$ unless j and k both exceed n_i. Hence the subspace spanned by

$$e_{n_{i+1}+1},\ldots, e_{n_i},$$

that is, by the basic elements of $\mathbf{B}(i)$ which are not in $\mathbf{B}(i+1)$, is contained in the subspace spanned by

$$e_j e_k \quad (j = n_i+1,\ldots, n_h;\ k = n_i+1,\ldots, n).$$

Hence, if
$$e_j e_k = \sum_{u=n_{i+1}+1}^{n_i} a_{jk}^u e_u + \sum_{u=1}^{n_{i+1}} b_{jk}^u e_u$$
$$(j = n_i+1,\ldots, n_h;\ k = n_i+1,\ldots, n),$$
the rank of the matrix (a_{jk}^u), where u denotes the column and jk the row, is $n_i - n_{i+1}$. Therefore this matrix contains a non-singular submatrix of order $n_i - n_{i+1}$. We can number the rows of (a_{jk}^u) so that

$$(a_v^u) \quad (v = n_{i+1}+1,\ldots, n_i)$$

is such a non-singular submatrix. Then the elements f_v defined by

$$f_v = \sum_{u=n_{i+1}+1}^{n_i} a_v^u e_u + \sum_{u=1}^{n_{i+1}} b_v^u e_u$$

are linearly independent. By definition they are products of the basic elements $e_{n_i+1},\ldots, e_n$. Moreover they are linearly dependent on $e_1,\ldots, e_{n_i}$ only. Therefore

$$e_1,\ldots, e_{n_{i+1}}, f_{n_{i+1}+1},\ldots, f_{n_i}, e_{n_i+1},\ldots, e_n$$

is a canonical basis for $\mathbf{A}$.

Thus the lemma is proved in the case when $\mathbf{B}(i)$ is a right power of $\mathbf{A}$. If $\mathbf{B}(i)$ is a left power of $\mathbf{A}$, the lemma can be proved in a similar manner.

3. The main theorems

Let $\mathbf{C}$ denote the greater of the two subspaces $\mathbf{A}^r$, $\mathbf{A}_s$. Then $\mathbf{C} = \mathbf{A}^r$ if $\mathbf{A}_s \leqslant \mathbf{A}^r$, and $\mathbf{C} = \mathbf{A}_s$ if $\mathbf{A}^r \leqslant \mathbf{A}_s$.

THEOREM 1. *Let* $\mathbf{A}$ *be a linear algebra of dimension* n *satisfying* (α), *and such that the derived algebra* $\mathbf{A}^{(1)}$ *is not identical with* $\mathbf{A}$. *If* $e_1,\ldots, e_n$ *is a basis for* $\mathbf{A}$ *such that* $e_1,\ldots, e_{n_2}$ *is a basis for* $\mathbf{A}^{(1)}$, *then, by a transformation of* $e_1,\ldots, e_{n_2}$, *a basis can be obtained such that the basic elements which are not in* $\mathbf{C}$ *are generated by* $e_{n_2+1},\ldots, e_n$.

Proof. By a suitable transformation of $e_1,\ldots, e_{n_2}$ we can obtain a canonical basis $e_1^*,\ldots, e_{n_2}^*, e_{n_2+1},\ldots, e_n$ for $\mathbf{A}$. Suppose that $\mathbf{C} = \mathbf{B}(q)$.

Then the subspaces $\mathbf{B}(2),\ldots,\mathbf{B}(q-1)$ satisfy the conditions of the lemma. Hence we can find a canonical basis

$$e_1^*,\ldots,e_{n_q}^*,f_{n_q+1},\ldots,f_{n_2},e_{n_2+1},\ldots,e_n$$

such that each element f_i $(i = n_q+1,\ldots,n_2)$ is linearly dependent on products of the succeeding elements. Theorem 1 is therefore proved.

THEOREM 2. *Let* $\mathbf{A}$ *be a nilpotent linear algebra of dimension* n *satisfying* (α). *If* $e_1,\ldots,e_n$ *is a basis for* $\mathbf{A}$ *such that* $e_1,\ldots,e_{n_2}$ *is a basis for* $\mathbf{A}^{(1)}$, *then* $\mathbf{A}$ *is generated by* $e_{n_2+1},\ldots,e_n$. *The nullity of* $\mathbf{A}$ *is* $n-n_2$ *and its genus is* n_2.

Proof. If $\mathbf{A}$ is nilpotent, $\mathbf{A}^r = 0 = \mathbf{A}_s$, and so $\mathbf{C} = 0$. Also $\mathbf{A}^{(1)}$ is not identical with $\mathbf{A}$ unless $\mathbf{A}$ itself is zero. Therefore, by Theorem 1, $e_1,\ldots,e_{n_2}$ can be replaced by basic elements linearly dependent on products of $e_{n_2+1},\ldots,e_n$. Hence $\mathbf{A}$ is generated by $e_{n_2+1},\ldots,e_n$. It follows that the nullity of $\mathbf{A}$ is at most $n-n_2$. Any element of $\mathbf{A}$ which is not in $\mathbf{A}^{(1)}$ cannot be obtained by forming sums of products of elements of $\mathbf{A}$. Since there are $n-n_2$ linearly independent elements of $\mathbf{A}$ which are not in $\mathbf{A}^{(1)}$, it follows that the nullity of $\mathbf{A}$ is at least $n-n_2$. Combining this with the previous result we see that the nullity of $\mathbf{A}$ is $n-n_2$; hence the genus is n_2. Theorem 2 is therefore proved.

Many familiar linear algebras satisfy (α). For example, in associative, commutative, and anti-commutative linear algebras (including Lie algebras), we have $\mathbf{A}^k = \mathbf{A}_k$ for each integer k, and so (α) is satisfied. An example of an algebra satisfying (α), but not the stronger condition $\mathbf{A}^k = \mathbf{A}_k$ for each integer k, is given by the multiplication table

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	0	0	0	0	0
e_2	0	0	0	0	0	0
e_3	0	0	0	0	0	0
e_4	0	0	0	0	0	e_1
e_5	0	0	e_1	e_3	e_2	e_4
e_6	0	0	e_2	0	0	e_5

In this case $\mathbf{A}^5 = 0 = \mathbf{A}_5$, and so $\mathbf{A}$ is nilpotent. The conditions assumed in Theorem 2 are all satisfied, and $\mathbf{A}$ is generated by e_6. However, $\mathbf{A}_3 < \mathbf{A}^3$ and $\mathbf{A}^4 < \mathbf{A}_4$.

An example of a nilpotent linear algebra which does not satisfy (α) is given by the table

	e_1	e_2	e_3
e_1	.	.	.
e_2	.	.	e_1
e_3	e_2	.	.

In this case $\mathbf{A}^4 = 0 = \mathbf{A}_4$, but $\mathbf{A}^3$ has basis e_1 and $\mathbf{A}_3$ has basis e_2. This algebra is not generated by e_3 although e_1, e_2, e_3 is a basis for $\mathbf{A}$ such that e_1, e_2 is a basis for $\mathbf{A}^{(1)}$. Thus the first part of Theorem 2 is not satisfied. However, $\mathbf{A}$ is generated by the single element $e_1+e_2+e_3$, so that the last part of Theorem 2 remains true, namely that the genus of $\mathbf{A}$ is equal to the dimension of $\mathbf{A}^{(1)}$. It remains open to conjecture whether this result is valid for any nilpotent linear algebra.

4. Corollaries

COROLLARY 1.† *If* $\mathbf{A}$ *is a nilpotent linear algebra of dimension* n (> 1) *such that* $x^2 = 0$ *for all* $x \in \mathbf{A}$, *then the dimension of* $\mathbf{A}^{(1)}$ *cannot exceed* $n-2$.

Proof. Since $x^2 = 0$ for all $x \in \mathbf{A}$, $\mathbf{A}$ satisfies the anti-commutative law $yz = -zy$. Therefore (α) is satisfied. If $\mathbf{A}^{(1)}$ is of dimension $n-1$, then, by Theorem 2, $\mathbf{A}$ is generated by a single element w. But the space generated by w is one-dimensional since $w^2 = 0$. Hence, if $n > 1$, the dimension of $\mathbf{A}^{(1)}$ cannot exceed $n-2$.

COROLLARY 2. *If* $\mathbf{A}$ *is a nilpotent associative algebra of dimension* n *such that the dimension of* $\mathbf{A}^{(1)}$ *is* $n-1$, *then* $\mathbf{A}$ *is isomorphic with the algebra of matrices of the form*

$$\begin{bmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 & . & . & . & \alpha_{n-1} & \alpha_n \\ 0 & 0 & \alpha_1 & \alpha_2 & . & . & . & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 0 & 0 & \alpha_1 & . & . & . & \alpha_{n-3} & \alpha_{n-2} \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \end{bmatrix}.$$

Proof. By Theorem 2, $\mathbf{A}$ is generated by a single element x. Therefore, since $\mathbf{A}$ is associative, every element is of the form

$$\alpha_1 x + \alpha_2 x^2 + \dots.$$

Since $\mathbf{A}$ is nilpotent, $\mathbf{A}^{n+1} = 0$, and so $x^{n+1} = 0$. Also, the elements $x, \dots, x^n$ are linearly independent since $\mathbf{A}$ is of dimension n. Therefore every element of $\mathbf{A}$ is of the form

$$\alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n,$$

where $x, x^2, \dots, x^n$ are all non-zero but $x^k = 0$ $(k \geqslant n+1)$. Corollary 2

† This result has been proved elsewhere (4) by a different method.

follows at once by identifying x with the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 1 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \end{bmatrix}$$

of order $n+1$.

COROLLARY 3. *If* $\mathbf{L}$ *is a solvable Lie algebra over a field of characteristic zero, and if the dimensions of* $\mathbf{L}^{(1)}$ *and* $\mathbf{L}^{(2)}$ *are* p *and* q *respectively, then the genus* γ *of* $\mathbf{L}$ *satisfies*

$$q \leqslant \gamma \leqslant p.$$

Proof. If $\mathbf{L}$ is a solvable Lie algebra over a field of characteristic zero, then $\mathbf{L}^{(1)}$ is nilpotent (**2**). Therefore, by Theorem 2, the nullity of $\mathbf{L}^{(1)}$ is $p-q$. Since there are $n-p$ linearly independent elements of $\mathbf{L}$ which are not in $\mathbf{L}^{(1)}$, it follows that the nullity of $\mathbf{L}$ is at most

$$(n-p)+(p-q) = n-q$$

and is at least $n-p$. Hence $q \leqslant \gamma \leqslant p$.

The genus γ can attain the maximum value p given by Corollary 3, as shown by Theorem 2, which implies that a nilpotent Lie algebra is of genus p (a nilpotent linear algebra is, of course, always solvable). An example of a solvable non-nilpotent Lie algebra for which $\gamma = p$ is given by the table

	e_1	e_2	e_3	e_4
e_1	.	.	e_1	$-e_2$
e_2	.	.	e_2	e_1
e_3	$-e_1$	$-e_2$	.	e_2
e_4	e_2	$-e_1$	$-e_2$	.

In this case $\gamma = 2 = p$.

The genus γ can attain the minimum value q given by Corollary 3. For example in the Lie algebra $\mathbf{L}$ defined by [see (**3**)]

$$e_i e_n = -e_n e_i = e_i \quad (i = 1,\ldots, n-1),$$

$$e_i e_j = 0 \quad (i,j = 1,\ldots, n-1),$$

$$e_n^2 = 0,$$

we have $q = 0$, since $\mathbf{L}^{(2)} = 0$, and $\gamma = 0$.

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ON THE L_p-CONVERGENCE OF EIGENFUNCTION EXPANSIONS

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1. Introduction†

LET b denote either a positive real number, or $+\infty$. Let $q(x)$ be a function continuous over the right half-line, λ a complex number, and let $\phi(x,\lambda)$, $\chi_b(x,\lambda)$ be the solutions of the differential equation

$$(D^2+\lambda-q(x))y = 0,$$

which satisfy, respectively,

$$\phi(0,\lambda) = \sin\alpha,$$
$$\phi'(0,\lambda) = -\cos\alpha,$$

and, in the Sturm–Liouville case (i.e. $b < \infty$),

$$\chi_b(b,\lambda) = \sin\beta,$$
$$\chi_b'(b,\lambda) = -\cos\beta.$$

In connexion with such solutions, primes denote partial differentiation with respect to x.

For finite b, let

$$\omega_b(\lambda) = \phi\chi_b'-\chi_b\phi',$$

and let $k_b(t)$ be the linear semi-continuous step function vanishing at $t = 0$ which increases by

$$\frac{a_{b,t}}{\pi\omega'(t)}$$

at each eigenvalue t, i.e. at each value of λ for which $\omega_b(\lambda) = 0$, and hence $\chi_b = a_{b,\lambda}\phi$. Then, if in particular $q \in L(0,\infty)$, $k_b(t)$ tends to a unique limit $k_\infty(t) = k(t)$ as $b \to \infty$, k being a function of bounded variation over $(-\infty,\infty)$.

Let $L_{p,b}$, $\mathscr{L}_p(b)$ be the spaces of real-valued functions of a real variable, f and F, for which

$$\|f\|_{p,b} = \left(\int\limits_0^b |f|^p\,dx\right)^{1/p} < \infty,$$

$$\left(\int\limits_{-\infty}^{\infty} |F|^p\,dk_b(t)\right)^{1/p} < \infty,$$

respectively; further, set $L_p = L_{p,\infty}$, $\mathscr{L}_p = \mathscr{L}_p(\infty)$, $\|f\|_p = \|f\|_{p,\infty}$.

† Unless other references are given, the results used in this paragraph are contained in, or follow immediately from the discussion of the Sturm–Liouville case in (**3**), Chapter 1, and the discussion of $k_b(t)$ and $k(t)$ given in (**2**), § **4**.

Henceforth we assume that $1 < p \leqslant 2$, and that $p^{-1}+q^{-1} = 1$. With the foregoing definitions, it is known that the equations

$$\Phi_b f = \int_0^b f(x)\phi(x,t)\,dx \quad (b < \infty),$$

$$\Psi_{\omega,b} f = \int_{-\omega}^{\omega} F(t)\phi(x,t)\,dk_b(t)$$

define transforms from $L_{2,b}$ onto $\mathscr{L}_2(b)$ in the one case, and from a subset of $\mathscr{L}_2(b)$ into $L_{2,b}$ in the other; further, if $f \in L_2$, or $F \in \mathscr{L}_2(b)$, $\Phi_b f$ converges in $\mathscr{L}_2$ to a function $\Phi_\infty f = \Phi f$ as $b \to \infty$, $\Psi_{\omega,b} f$ converges in L_2 to a function $\Psi_b F$ as $\omega \to \infty$; and Φ_b, Ψ_b are adjoint unitary transforms between $L_{2,b}$ and $\mathscr{L}_2(b)$.

Now ϕ is bounded, uniformly in all eigenvalues λ, over each $[0,b]$; it follows that for $b < \infty$, at all eigenvalues λ, we have

$$|\Phi_b f| \leqslant A_b \|f\|_{1,b},$$

where A_b is a constant depending only on b; hence, by the Hausdorff–Young theorem† the closure of Φ_b in $L_{p,b} \times \mathscr{L}_p(b)$ is a bounded transform from $L_{p,b}$ into $\mathscr{L}_q(b)$, for each finite b $(1 < p \leqslant 2)$; the situation in the case $b = \infty$ has yet to be clarified.

A further question as yet unresolved is the following: does the closure of Ψ_b in $L_{p,b} \times \mathscr{L}_q(b)$ constitute the transform inverse to Φ_b on $\Phi_b(L_{p,b})$? Or put differently, if $f \in L_{p,b}$ and $\Theta_{\omega,b} = \Psi_{\omega,b}\Phi_b$, does $\Theta_{\omega,b} f$ converge to f in $L_{p,b}$ as $\omega \to \infty$? (It turns out that $\Theta_{\omega,b} f$ can be defined in the limiting case without knowing whether or not Φ is defined.) In the case of Fourier series and transforms, where $q(x) = 0$, $\sin\alpha$ and $\sin\beta$ are both equal to 0 or 1, this question is answered in the affirmative [(**5**), 153 and 318] by the use of the Riesz–Stein theorem on conjugate functions [(**5**), 147–9]. The present paper is an attempt to dispose of the question in the Sturm–Liouville case, and to find sufficient conditions on $q(x)$ to provide an affirmative answer in the singular case; the conditions used are

(i) $xq(x)$ integrable,

(ii) $xq(x)$ of bounded variation

over the half-line $x \geqslant 0$.

2. A lemma concerning transformations in $L_{p,b}$

LEMMA A. *Let $f \to \Theta_\omega f$ be a linear transformation depending on the parameter ω, from $L_{p,b}$ into itself, such that*

$$\|\Theta_\omega f\|_{p,b} \leqslant A\|f\|_{p,b}, \tag{2.1}$$

† See, e.g. (**1**). (In this paper, the theorem is formulated for arbitrary measure spaces.)

where A is a constant independent of f and ω, and

$$\|f-\Theta_\omega f\|_{p,b} \to 0 \quad as\ \omega \to \infty, \tag{2.2}$$

on a dense subset of $L_{p,b}$.

Then

$$\|f-\Theta_\omega f\|_{p,b} \to 0 \quad as\ \omega \to \infty$$

on $L_{p,b}$, $b = \infty$ being permissible.

Let $f \in L_{p,b}$; for any $\epsilon > 0$, there exist f_1, f_2 such that $f = f_1+f_2$, and

$$\|f_2\|_{p,b} < \frac{\epsilon}{3(1+A)}, \qquad \|f_1-\Theta_\omega f_1\|_{p,b} \to 0.$$

It follows that there exists ω_0 such that, for all $\omega > \omega_0$,

$$\begin{aligned}\|f-\Theta_\omega f\|_{p,b} &< \|f_1-\Theta_\omega f_1\|_{p,b}+\|f_2\|_{p,b}+\|\Theta_\omega f_2\|_{p,b}\\ &< \epsilon,\end{aligned}$$

i.e. $\Theta_\omega f \to f$ in $L_{p,b}$.

THE STURM–LIOUVILLE CASE

($0 < b < \infty$; $q(x)$ *continuous for* $0 \leqslant x \leqslant b$)

3. An inequality concerning Fourier cosine series

Define

$$j_{m,b}(x,y) = 2b^{-1}\sum_{k=0}^{m} \cos kx \cos ky,$$

$$J_{m,b}(f,g) = \int_0^b\int_0^b f(x)g(y)j_{m,b}(x,y)\,dxdy.$$

LEMMA B. *There exists a number B_p depending only on p such that*

$$|J_{m,b}| \leqslant B_p\|f\|_{p,b}\|g\|_{q,b}$$

whenever $f \in L_{p,b}$ and $g \in L_{q,b}$.

It is known from the theory of the conjugate function [(5), 147–9] that

$$\left|P\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{f(x)g(y)}{x-y}\,dxdy\right| \leqslant B_p\left(\int_{-\pi}^{\pi}|f|^p\,dx\right)^{1/p}\left(\int_{-\pi}^{\pi}|g|^q\,dx\right)^{1/q},$$

where P denotes the principal value of the integral, and the left-hand side exists whenever the right-hand side is defined (and finite). It is readily shown that a similar inequality, with a slightly different constant B_p, holds good if the $x-y$ occurring in the denominator of the integrand

on the left is replaced by either $\sin\frac{1}{2}(x+y)$ or $\sin\frac{1}{2}(x-y)$; hence, on replacing f and g by functions such as

$$f_1(x) = \begin{cases} f(x)\sin(ax+b) & (0 \leqslant x \leqslant \pi), \\ 0 & (-\pi \leqslant x < 0), \end{cases}$$

and $g_1(x)$ similarly defined in relation to g, we obtain

$$\left| P\int_0^\pi\int_0^\pi \frac{f(x)g(y)\sin(ax+b)\sin(cy+d)}{\sin\frac{1}{2}(x+y)}\,dxdy \right| \leqslant B_p\|f\|_{p,\pi}\|g\|_{q,\pi}. \tag{3.1}$$

Now
$$j_{m,\pi} = \frac{1}{\pi}\sum_{k=0}^{m} \{\cos k(x+y)+\cos k(x-y)\}$$
$$= \frac{1}{2\pi}\left\{\frac{\sin(m+\frac{1}{2})(x+y)}{\sin\frac{1}{2}(x+y)} + \frac{\sin(m+\frac{1}{2})(x-y)}{\sin\frac{1}{2}(x-y)} + 2\right\};$$

the desired inequality for $J_{m,\pi}(f,g)$ is therefore a consequence of (3.1) and the relation

$$\left| \int_0^\pi\int_0^\pi fg\,dxdy \right| \leqslant \|f\|_{1,\pi}\|g\|_{1,\pi} \leqslant \pi\|f\|_{p,\pi}\|g\|_{q,\pi}.$$

For $b \neq \pi$, we note that the transformation $x \to b\pi^{-1}x$, $y \to b\pi^{-1}y$ carries $J_{m,b}(f,g)$ into $b\pi^{-1}J_{m,\pi}(f_b, g_b)$, where

$$f_b(x) = f(b\pi^{-1}x), \qquad g_b(x) = g(b\pi^{-1}x);$$

hence
$$J_{m,b}(f,g) \leqslant B_p\frac{b}{\pi}\|f_b\|_{p,\pi}\|g_b\|_{q,\pi}$$
$$= B_p\|f\|_{p,b}\|g\|_{q,b}.$$

4. Results concerning the Sturm–Liouville theory

Let b be fixed, $0 < b < \infty$, and suppose that $\sin\alpha\sin\beta \neq 0$.

Let
$$G(x,y,\lambda) = \begin{cases} \dfrac{\chi_b(x,\lambda)\phi(y,\lambda)}{2\pi\omega_b(\lambda)} & (0 \leqslant x < y), \\[2ex] \dfrac{\chi_b(y,\lambda)\phi(x,\lambda)}{2\pi\omega_b(\lambda)} & (y \leqslant x), \end{cases}$$

$$h_{\omega,b}(x,y) = \int_{-\omega}^{\omega} \phi(x,t)\phi(y,t)\,dk_b(t),$$

$$\Theta_{\omega,b}f(y) = \int_0^b\int_0^b f(x)h_{\omega,b}(x,y)\,dxdy = \Psi_{\omega,b}\,\Phi_b f(y).$$

For each b, there exists $\omega = \omega_b$ such that $\omega_b(t) \neq 0$ for $t < -\omega_b$ [(**3**), 12]; for $\omega > \omega_b$, define $C_\omega = C_{\omega,b}$ to be the well-known contour [(**3**), 13] symmetrical about the real axis, which corresponds, in the upper half of the λ-plane, to the boundary of the quarter-square in the s-plane

$$s = \begin{cases} \sqrt{\omega^*}+i\tau & (0 \leqslant \tau \leqslant \sqrt{\omega^*}), \\ \sigma+i\sqrt{\omega^*} & (0 \leqslant \sigma \leqslant \sqrt{\omega^*}), \end{cases}$$

where $\lambda = s^2$, $s = \sigma+i\tau$, and ω^* is the point bisecting the interval between the greatest eigenvalue not exceeding ω and the succeeding one.

Then by [(**3**), 13]

$$\frac{1}{2\pi i}\int\limits_{C_\omega} G(x,y,\lambda) = h_{\omega,b}(x,y)$$

and

$$G(x,y,\lambda) = O\left(\frac{e^{-\tau|y-x|}}{|\lambda|}\right)+\begin{cases} \dfrac{\cos\{s(b-x)\}\cos sy}{s\sin sb} & (0 \leqslant x \leqslant y), \\ \dfrac{\cos\{s(b-y)\}\cos sx}{s\sin sb} & (y < x). \end{cases}$$

Hence, by the calculus of residues,

$$h_{\omega,b}(x,y) = j_{m,b}(x,y)+O\left\{\int\limits_{C_\omega} \frac{e^{-\tau|y-x|}}{|\lambda|}\,|d\lambda|\right\}, \tag{4.1}$$

where m is the greatest integer not exceeding $\sqrt{\omega}$. The second term on the right of this expression is

$$O\left(\int\limits_{\tau=0}^{\sqrt{\omega}} \frac{e^{-\tau|y-x|}}{\sqrt{\omega}}\,d\tau\right)+O\left(\int\limits_{\sigma=0}^{\sqrt{\omega}} \frac{e^{-\sqrt{\omega}|y-x|}}{\sqrt{\omega}}\,d\sigma\right)$$

$$= O\left(\frac{1-e^{-\sqrt{\omega}|y-x|}}{\sqrt{\omega}\,|y-x|}\right)+O(e^{-\sqrt{\omega}|y-x|}); \tag{4.2}$$

in particular then,

$$h_{\omega,b}(x,y) = j_{m,b}(x,y)+O(1). \tag{4.3}$$

THEOREM I. *The Sturm–Liouville expansion of a function of class $L_{p,b}$ converges in mean to the function, i.e.*

$$\lim_{\omega\to\infty} \|\Theta_{\omega,b}f-f\|_{p,b} = 0, \tag{4.4}$$

and further
$$\|\Theta_{\omega,b}f\|_{p,b} \leqslant A_{p,b}\|f\|_{p,b}, \tag{4.5}$$

where $A_{p,b}$ does not depend on the function f.

If $\sin\alpha\sin\beta \neq 0$, and $f \in L_{q,b}$, we have, using (4.3) and then Lemma B,

$$\int_0^b (\Theta_{\omega,b}f)g\,dy = \int_0^b\int_0^b f(x)g(y)h_{\omega,b}(x,y)\,dxdy$$

$$= J_{m,b}(f,g)+O\left(\int_0^b\int_0^b fg\,dxdy\right)$$

$$\leqslant B_p\|f\|_{p,b}\|g\|_{q,b}+C\|f\|_{1,b}\|g\|_{1,b}$$

$$\leqslant (B_p+bC)\|f\|_{p,b}\|g\|_{q,b},$$

and this is equivalent to (4.5).

We have now shown that $\Theta_{\omega,b}$ satisfies the condition (2.1) of Lemma A; further, (4.4) is known to be true for $p = 2$, $L_{2,b}$-convergence implies $L_{p,b}$-convergence, and $L_{2,b}$ is dense in $L_{p,b}$ $(1 < p \leqslant 2)$, so that the condition (2.2) of Lemma A is satisfied, and (4.4) follows.

A similar argument applies if $\sin\alpha\sin\beta = 0$.

THE SINGULAR CASE ($b = \infty$)

5. Additional hypotheses and notation

Let $q^*(x) = xq(x)$; we now assume that in addition to $q(x)$ being continuous,

$$q^* \in L(0,\infty) \tag{5.1}$$

and

$$q^* \text{ is of bounded variation on } (0,\infty). \tag{5.2}$$

With these assumptions it follows that $q \in L(0,\infty)$, and also

$$\int_x^\infty |q(t)|\,dt = O\left\{\frac{1}{1+x}\right\} \quad \text{as } x \to \infty. \tag{5.3}$$

Further, it is known† that

$$k(t) = \frac{1}{\pi}\int_0^t \frac{du}{\sqrt{u\{\mu^2(u)+\nu^2(u)\}}},$$

where

$$\mu(t) = \sin\alpha - \frac{1}{s}\int_0^\infty \sin sy\, q(y)\phi(y,t)\,dy,$$

$$\nu(t) = -\frac{\cos\alpha}{s}+\frac{1}{s}\int_0^\infty \cos sy\, q(y)\phi(y,t)\,dy,$$

† (**3**), 99–101. Note, however, that the functions $k(t)$ used in (**3**) and (**2**) differ by the constant factor π^{-1}.

and $s^2 = t$. Set also

$$H_{c,d}(x,y) = \int_c^d \phi(x,t)\phi(y,t)\,dk(t). \tag{5.4}$$

6. Asymptotic expansions of various functions

The function $\phi(x,\lambda)$

It is known [(**3**), 9] that

$$\phi(x,\lambda) = \cos sx \sin\alpha - \frac{\sin sx \cos\alpha}{s} + \frac{1}{s}\int_0^x \sin s(x-y)q(y)\phi(y,\lambda)\,dy$$

$$= \cos sx \sin\alpha + O(s^{-1}), \tag{6.1}$$

uniformly in x, since $\phi = O(1)$ as $s \to \infty$ [(**3**), 98]. Further,

$$\phi(x,\lambda) = \cos sx \sin\alpha - \frac{\sin sx \cos\alpha}{s} +$$

$$+\frac{1}{s}\int_0^\infty \sin s(x-y)q(y)\phi(y,\lambda)\,dy + O\left(\int_x^\infty |q(y)|\,dy\right)$$

$$= \mu(\lambda)\cos sx + \nu(\lambda)\sin sx + O\left\{\frac{1}{1+x}\right\}. \tag{6.2}$$

If, however, $\sin\alpha = 0$, then (6.2) must be replaced by

$$\phi(x,\lambda) = -\frac{\sin sx \cos\alpha}{s} + O\left(\frac{1}{s^2}\right) = O\left(\frac{1}{s}\right). \tag{6.3}$$

The function $\partial\phi(x,\lambda)/\partial s$

Now

$$\frac{\partial\phi}{\partial s} = -x\sin sx \sin\alpha - \frac{x\cos sx\cos\alpha}{s} + \frac{\sin sx\cos\alpha}{s^2} -$$

$$-\frac{1}{s^2}\int_0^x \sin s(x-y)q(y)\phi(y,\lambda)\,dy + \frac{1}{s}\int_0^x (x-y)\cos s(x-y)q(y)\phi(y,\lambda)\,dy +$$

$$+\frac{1}{s}\int_0^x \sin s(x-y)q(y)\frac{\partial\phi}{\partial s}\,dy = O(x) + O\left\{\frac{1}{s}\int_0^x \left|q(y)\frac{\partial\phi}{\partial s}\right|dy\right\}. \tag{6.4}$$

Hence, if $M_x = \sup \dfrac{1}{1+y}\left|\dfrac{\partial\phi}{\partial s}\right|$ for $0 \leqslant y \leqslant x$, we have, as $s \to \infty$,

$$(1+x)M_x = O(1+x) + O\left(\frac{(1+x)M_x}{s}\right)$$

and
$$M_x = \frac{O(1)}{1+O(s^{-1})} = O(1),$$

uniformly in x; i.e. as $x, s \to \infty$,

$$\frac{\partial \phi}{\partial s} = O(1+x). \tag{6.5}$$

Also
$$\frac{\partial \phi}{\partial s} = -x \sin sx \sin \alpha + O\left\{\frac{1+x}{s}\right\} \tag{6.6}$$

by (6.4) and (6.5).

Similarly, if $\sin \alpha = 0$,

$$\frac{\partial \phi}{\partial s} = O\left(\frac{1+x}{s}\right) = -\frac{x \cos sx}{s} + O\left\{\frac{1+x}{s^2}\right\}. \tag{6.7}$$

The functions $\mu(\lambda)$, $\dfrac{d}{ds}\mu(\lambda)$

We have
$$\mu(\lambda) = \sin \alpha - \frac{1}{s} \int_0^\infty \sin sy\, q(y)\phi(y,\lambda)\, dy = O(1) \tag{6.8}$$

as $s \to \infty$. For $\sin \alpha \neq 0$,

$$\frac{1}{\mu(\lambda)} = O(1) \tag{6.9}$$

and, for $\sin \alpha = 0$, using (6.3) we have

$$\mu(\lambda) = O(s^{-2}). \tag{6.10}$$

Further

$$\frac{d\mu}{ds} = \frac{1}{s^2} \int_0^\infty \sin sy\, q(y)\phi(y,\lambda)\, dy - \frac{1}{s} \int_0^\infty \cos sy\, yq(y)\phi(y,\lambda)\, dy + $$

$$+ \frac{1}{s} \int_0^\infty \sin sy\, q(y) \frac{\partial}{\partial s}\phi(y,\lambda)\, dy$$

$$= O\left(\frac{1}{s^2}\right) + \frac{\sin \alpha}{s} \int_0^\infty q^*(y)(\sin^2 sy - \cos^2 sy)\, dy,$$

using (5.3), (6.3), and (6.6); and, since the second term on the right is the Fourier transform of a function of bounded variation,

$$\frac{d\mu}{ds} = O(s^{-2}). \tag{6.11}$$

Similarly, if $\sin\alpha = 0$,

$$\frac{d\mu}{ds} = O(s^{-3}). \tag{6.12}$$

The functions $\nu(\lambda)$, $\dfrac{d}{ds}\nu(\lambda)$

Now
$$\nu(\lambda) = -\frac{\cos\alpha}{s}+\frac{1}{s}\int\limits_0^\infty \cos sy\, q(y)\phi(y,\lambda)\, dy$$
$$= O(s^{-1}).$$

Also, if $\sin\alpha = 0$, $\quad \nu(\lambda) = -s^{-1}+O(s^{-2})$,

$$\frac{1}{\nu(\lambda)} = O(s). \tag{6.13}$$

On differentiating (6.13) and substituting from (6.5), we obtain further

$$\frac{d}{ds}\nu(\lambda) = O\left(\frac{1}{s}\right). \tag{6.14}$$

The function $k(\lambda)$

For $\lambda = s^2$, $s \geqslant 0$,

$$k(\lambda) = \frac{2}{\pi}\int\limits_0^s \frac{du}{\mu^2(u^2)+\nu^2(u^2)} = \frac{2}{\pi}\int\limits_0^s l(u)\, du, \tag{6.15}$$

where
$$l(u) = \frac{1}{\mu^2(u^2)+\nu^2(u^2)}.$$

Then
$$l'(u) = -\frac{2\mu\dfrac{d\mu}{ds}+2\nu\dfrac{d\nu}{ds}}{(\mu^2+\nu^2)^2}$$
$$= \begin{cases} O(s^{-2}) & \text{if } \sin\alpha \neq 0, \\ O(s) & \text{if } \sin\alpha = 0, \end{cases} \tag{6.16}$$

where we use, in the first case, the results (6.8), (6.9), (6.11), and (6.14) giving asymptotic relations for μ, ν and their derivatives, and for μ^{-1}, and, in the second case, the similar results (6.10), (6.13), (6.14) on μ, ν and their derivatives, and the result (6.12) on ν^{-1}. Combining (6.16) with the results just referred to, we obtain fairly easily the following lemma.

LEMMA C. *There exists a number $s_0 > 0$ such that the functions $\nu^2 l$, $\mu\nu l$, and $\mu^2 l$ are of bounded variation on (s_0, ∞).*

7. The integrals $P\int\int \frac{f(x)g(y)}{x\pm y}\,dxdy$

Let Q be the positive quadrant of the (x,y)-plane, R the closed region of Q bounded by the lines

$$y = (1/\sqrt{3})x, \qquad y = \sqrt{3}\,x.$$

Let

$$S = Q-R,$$

$$T = E\{\theta;\, 0 \leqslant \theta \leqslant \tfrac{1}{6}\pi \text{ or } \tfrac{1}{3}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi\}.$$

The notation $z = O(\zeta)$ will mean, in this paragraph, that if z and ζ are functions of p $(1 < p \leqslant 2)$, there exists a constant B_p, depending only on p such that $|z| < B_p|\zeta|$, and, if, in addition, z and ζ are functions of s, the inequality will hold for all s exceeding some fixed, positive s_0.

Lemma D. *If $f \in L_p$, $g \in L_q$, then the integrals*

$$\iint\limits_R \frac{f(x)g(y)}{x\pm y}\,dxdy, \qquad \iint\limits_R \frac{f(x)g(y)}{x+1}\,dxdy, \qquad \iint\limits_R \frac{f(x)g(y)}{y+1}\,dxdy$$

are all $O(\|f\|_p\|g\|_q)$.

We consider first

$$\begin{aligned}\iint\limits_S \frac{fg}{x-y}\,dxdy &= O\left(\int\limits_T\int\limits_0^\infty \frac{f(r\cos\theta)g(r\sin\theta)r\,drd\theta}{r|\cos\theta-\sin\theta|}\right)\\ &= O\left(\int\limits_T \|f(r\cos\theta)\|_p\,\|g(r\sin\theta)\|_q\,d\theta\right)\\ &= O\left(\|f\|_p\|g\|_q\int\limits_T \frac{d\theta}{|\cos\theta|^{1/p}|\sin\theta|^{1/q}}\right)\\ &= O(\|f\|_p\|g\|_q), \quad \text{since } p^{-1}, q^{-1} < 1.\end{aligned}$$

The corresponding result, with Q replacing S as range of integration, is known [(5), 318]; hence the result for R follows by subtraction. The remaining results stated in the lemma can be obtained by the type of argument used above for

$$\iint\limits_S \frac{fg}{x-y}\,dxdy,$$

the point to notice being that, on R,

$$\frac{1}{1+x}, \qquad \frac{1}{1+y}, \qquad \frac{1}{x+y}$$

are all $O(r^{-1})$.

LEMMA E. *If $f \in L^p$, $g \in L^q$, and $m(t)$ is of bounded variation on (a, ∞) $(a > 0)$, then, for $d > a$,*

$$\iint_R f(x)g(y) \int_a^d \cos xt \cos yt \, m(t) \, dt \, dxdy = O(\|f\|_p \|g\|_q), \tag{7.1}$$

and the result holds if either $\cos xt$ *or* $\cos yt$, *or both, be replaced by* $\sin xt$, $\sin yt$, *respectively.*

Let

$$\begin{aligned} j(t) &= 2\int_a^t \cos xu \cos yu \, du \\ &= \int_a^t \{\cos(x+y)u + \cos(x-y)u\} \, du \\ &= \left[\frac{\sin(x+y)u}{x+y} + \frac{\sin(x-y)u}{x-y}\right]_a^t \\ &= O\left(\frac{1}{x+y}\right) + \frac{\sin\{(x-y)t\} - \sin\{(x-y)a\}}{x-y}. \end{aligned}$$

Hence by applying Lemma D to the functions

$$f(x)\sin xu, \quad g(y)\cos yu, \quad \text{etc.,} \quad \text{with } u = t \text{ or } u = a,$$

we see that

$$\iint_R f(x)g(y)j(t) \, dxdy = O(\|f\|_p \|g\|_q).$$

Also, since

$$\int_a^d \cos xt \cos yt \, m(t) \, dt = [m(t)j(t)]_a^d - \int_a^d j(t) \, dm(t),$$

the expression on the left of (7.1) is equal to

$$\begin{aligned} O(\|f\|_p \|g\|_q) &- \iint_R f(x)g(y)\left\{\int_a^d j(t) \, dm(t)\right\} dxdy \\ &= O(\|f\|_p \|g\|_q) - \int_a^d \left\{\iint_R f(x)g(y)j(t) \, dxdy\right\} dm(t) \\ &= O(\|f\|_p \|g\|_q)\left\{1 + \int_a^d |dm(t)|\right\} \\ &= O(\|f\|_p \|g\|_q). \end{aligned}$$

The order of integration is interchangeable because all integrals converge absolutely.

This proves (7.1); and, if, say, $\cos xt$ is replaced by $\sin xt$, we have replacing $j(t)$ the function

$$-\left[\frac{\cos\{(x+y)u\}}{x+y}+\frac{\cos\{(x-y)u\}}{x-y}\right]_a^t,$$

and the argument proceeds in much the same way; similarly in the other cases.

LEMMA F. *There exists a number $c > 0$ such that, for any $\omega > c$, and $f \in L_p$, $g \in L_q$,*

$$\iint_R f(x)g(y)H_{c,\omega}(x,y)\,dxdy = O(\|f\|_p\|g\|_q).$$

On substituting from (5.4), (6.2) and (6.15), we have

$$H_{c,\omega}(x,y) = \int_c^\omega \phi(x,t)\phi(y,t)\,dk(t)$$

$$= \int_{\sqrt{c}}^{\sqrt{\omega}} \Big[\mu^2\cos sx\cos sy+\mu\nu\{\cos sx\sin sy+\sin sx\cos sy\}+$$

$$+\nu^2\sin sx\sin sy+O\Big(\frac{1}{1+x}\Big)+O\Big(\frac{1}{1+y}\Big)\Big]l(s)\,ds.$$

Since the functions $\mu^2 l$, $\mu\nu l$, $\nu^2 l$ are all of bounded variation over some (c,∞), by Lemma C, the result required follows from Lemmas D and E.

LEMMA G. *There exists a number $c > 0$, such that, for any $\omega > c$, and $f \in L_p$, $g \in L_q$,*

$$\iint_S fgH_{c,\omega}\,dxdy = O(\|f\|_p\|g\|_q). \tag{7.2}$$

Since $k(t) = \lim_{b\to\infty} k_b(t)$ [see § 1], it follows, from the Helly–Bray theorem [(4), 31–33], that

$$H_{c,\omega}(x,y) = \lim_{b\to\infty}\{h_{\omega,b}(x,y)-h_{c,b}(x,y)\},$$

and, since, as is easily verified, the relations (4.1), (4.2) hold uniformly in b provided that $q \in L(0,\infty)$, we have on S,

$$H_{-\omega,\omega}(x,y) = \int_{-\sqrt{\omega}}^{\sqrt{\omega}} \cos sx\cos sy\,ds+O\left\{\frac{1-e^{-\sqrt{\omega}|y-x|}}{\sqrt{\omega}|y-x|}\right\}+O(e^{-\sqrt{\omega}|y-x|})$$

$$= O\Big(\frac{1}{|y-x|}\Big). \tag{7.3}$$

A formula similar to (7.2), with $-\omega$ replacing c as the lower limit of integration therefore holds, by Lemma D. Now, for some $c > 0$, $k(\lambda) = 0$ for all $\lambda < -c$ [(3), 101]; hence, provided that $\omega > c$,

$$H_{-\omega,\omega} = H_{-c,c}+H_{c,\omega}.$$

Hence

$$\iint_S fgH_{c,\omega}\,dxdy = \iint_S fgH_{-\omega,\omega}\,dxdy - \iint_S fgH_{-c,c}\,dxdy$$
$$= O(\|f\|_p\|g\|_q).$$

LEMMA H. *There exists $c > 0$ such that, for any $\omega > c$, $f \in L_p$, $g \in L_q$,*

$$\iint_Q fgH_{c,\omega}\,dxdy = O(\|f\|_p\|g\|_q).$$

This follows immediately from Lemmas F and G.

8. THEOREM II. *Let $1 < p \leqslant 2$, $f \in L_p$, and $q(x)$ satisfy the conditions* (5.1) *and* (5.2). *Then*

$$\|\Theta_\omega f - f\|_p \to 0 \quad \textit{as } \omega \to \infty. \tag{8.1}$$

Further, for given $q(x)$ there exists a constant c and numbers A_p depending only on p such that

$$\|\Theta_\omega f\|_p \leqslant A_p\|f\|_p, \tag{8.2}$$

for all $\omega > c$.

Let
$$f_n(x) = \begin{cases} f(x) & (0 \leqslant x \leqslant n), \\ 0 & (x > n), \end{cases}$$

and let $g \in L_q \cap L_2$. By the L_2-theory applied to f_n and g,

$$\lim_{\omega\to\infty} \iint_Q f_n gH_{-\omega,\omega}\,dxdy = \int f_n(x)g(x)\,dx;$$

hence, by Hölder's inequality, there exists $\omega_0 = \omega_0(f_n, g)$, such that

$$\left|\iint_Q f_n gH_{-\omega,\omega}\,dxdy\right| \leqslant 2\|f_n\|_p\|g\|_q \quad (\omega > \omega_0)$$
$$\leqslant 2\|f\|_p\|g\|_q.$$

Also
$$\left|\iint_Q f_n gH_{c,\omega}\,dxdy\right| \leqslant B_p\|f\|_p\|g\|_q,$$

by Lemma H; by subtraction, therefore,

$$\left|\iint_Q f_n gH_{-\omega,c}\,dxdy\right| \leqslant (2+B_p)\|f\|_p\|g\|_q \tag{8.3}$$

for $\omega > \omega_0$; but, since $H_{-\omega,c} = H_{-c,c}$, this result is independent of ω_0, and so of f and g. On recombining (8.3) with Lemma H, we obtain

$$\left|\iint_Q f_n g H_{-\omega,\omega}\,dxdy\right| \leqslant (2+2B_p)\|f\|_p\|g\|_q,$$

provided only that $\omega > c$. Since g is an arbitrary function of $L_2 \cap L_p$, we have

$$\|\Theta_\omega f_n\|_p \leqslant A_p\|f\|_p \quad (\omega > c). \tag{8.4}$$

Now
$$\Theta_\omega f_n(y) = \int_0^\infty f_n(x) H_{-\omega,\omega}(x,y)\,dx;$$

since $f_n \to f$ in L_p-mean, and, by (7.3), $H_{-\omega,\omega} \in L_q$ as a function of x (or y) for fixed ω, we have from Hölder's inequality

$$\lim_{n\to\infty} \Theta f_n(y) = \Theta_\omega f(y) \quad (0 \leqslant y < \infty).$$

Fatou's lemma applied to (8.4) now yields (8.2).

In view of Lemma A and (8.2) it remains only to show that (8.1) holds on a dense subset of L_p. Suppose that f vanishes outside of $(0,\chi)$, and is bounded. Then, using (7.3), we have, for any $\delta > 0$,

$$\Theta_\omega f(y) = \frac{1}{2\pi}\int_0^\infty f(x)\left\{\frac{\sin\{\sqrt{\omega}(x-y)\}}{x-y} + \frac{\sin\{\sqrt{\omega}(x+y)\}}{x+y}\right\}dx +$$
$$+\begin{cases} O\left(\int_0^{y-\delta} + \int_{y+\delta}^{2\chi} \frac{dx}{\sqrt{\omega}\delta}\right) + O\left(\int_{y-\delta}^{y+\delta} dx\right) & (0 \leqslant y < 2\chi), \\ O\left(\frac{1}{\sqrt{\omega}|y-\chi|}\right) & (2\chi \leqslant y). \end{cases}$$

The first term on the right tends to f in L_p-mean, by the theory of Fourier transforms [(5), 318]; the norm of the second (in L_p) is

$$O\left(\frac{\chi^{1/p}}{\sqrt{\omega}\delta}\right) + O(\delta\chi^{1/p}) + O\left(\frac{1}{\sqrt{\omega}}\right)\left(\int_{2\chi}^\infty \frac{dy}{|y-\chi|^p}\right)^{1/p};$$

choosing first δ small and then ω large, we make this last expression arbitrarily small and the theorem follows.

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SOME FURTHER PATHOLOGICAL EXAMPLES IN THE THEORY OF DENUMERABLE MARKOV PROCESSES

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1. Introduction and summary

For our purposes we may identify a temporally homogeneous Markov process $\mathscr{P}$ having a countable infinity of states with the array $p_{ij}(t)$ $(i,j = 0,1,2,...;\, t \geqslant 0)$ of its transition probabilities, and these (when the grosser pathologies have been eliminated) may be axiomatized as follows:

$$\text{(I)} \qquad p_{ij}(t) \geqslant 0;$$

$$\text{(II)} \qquad \sum_{\alpha=0}^{\infty} p_{i\alpha}(t) = 1;$$

$$\text{(III)} \qquad p_{ij}(u+v) = \sum_{\alpha=0}^{\infty} p_{i\alpha}(u)p_{\alpha j}(v);$$

$$\text{(IV)} \qquad p_{ij}(0) = \delta_{ij};$$

$$\text{(V)} \qquad p_{ii}(t) \to 1 \quad (t \downarrow 0).$$

If (II) is replaced by the weaker requirement,

$$\text{(II*)} \qquad \sum_{\alpha=0}^{\infty} p_{i\alpha}(t) \leqslant 1,$$

then [following Jensen (**6**)] we shall refer to the system as a *quasi-process*. Now let l be the Banach space of absolutely convergent series

$$x \equiv (x_0, x_1, x_2,...)$$

(where the x's are real numbers) with norm

$$\|x\| \equiv \sum_{\alpha=0}^{\infty} |x_\alpha| < \infty,$$

and define a new element of l whose jth component is

$$(P_t x)_j \equiv \sum_{\alpha=0}^{\infty} x_\alpha p_{\alpha j}(t). \tag{1}$$

Then P_t is a transition operator (i.e. it is a positive operator mapping l linearly into itself, and is norm-preserving on the positive cone) and

(A) $$P_{u+v} = P_u P_v;$$

(B) $$P_0 = I;$$

(C) $$\|P_t x - x\| \to 0 \quad (t \downarrow 0);$$

that is, $\{P_t : t \geqslant 0\}$ is a transition semigroup $\mathscr{G}$ of operators on l. (All this holds also if the p-functions constitute a quasi-process, but then the operators P_t are merely norm-non-increasing, and we speak of *contraction* operators and a *contraction* semigroup.) Conversely every transition (contraction) semigroup has a unique representation (1) in which the p-functions constitute a process (quasi-process), so that for mathematical purposes we can identify a transition semigroup $\mathscr{G}$ with the associated process $\mathscr{P}$.

For a transition (or contraction) semigroup the quotient $(P_t x - x)/t$ converges strongly when $t \downarrow 0$ to a limit $\Omega x \in l$ if and only if x lies in a certain linear dense set $\mathscr{D}(\Omega)$. The semigroup is uniquely determined when Ω is given, and this operator is called the ***infinitesimal generator***. Also, if $\lambda > 0$ and if u^i is the ith unit coordinate vector in l, then the equation

$$\lambda\xi - \Omega\xi = u^i$$

has a unique solution $\xi^i(\lambda)$ in $\mathscr{D}(\Omega)$, and

$$(\xi^i(\lambda))_j = \int_0^\infty e^{-\lambda t} p_{ij}(t)\, dt \quad (\lambda > 0). \tag{2}$$

Now the p-functions are bounded and continuous, and so (2) determines them uniquely in virtue of Lerch's theorem. The passage from Ω to the p-functions thus involves the solution of a set of linear equations and the inversion of a sequence of Laplace transforms (the existence and uniqueness of the solution being guaranteed by the theory).

The differential behaviour at $t = 0$ has also been studied for the p-functions themselves. It is known [Doob (**2**), Kolmogorov (**10**), Kendall (**8**)] that the limits

$$q_{ij} \equiv \lim_{t \downarrow 0} (p_{ij}(t) - \delta_{ij})/t$$

always exist; they are finite if $i \neq j$, and (zeros apart) the elements of the q-matrix $\{q_{ij}\}$ are negative on and positive off the leading diagonal; lastly the semi-conservation condition,

$$\sum_{\alpha \neq i} q_{i\alpha} \leqslant -q_{ii}, \tag{3}$$

always holds. The inequality (3) asserts roughly, when $\mathscr{P}$ is an (honest†) process, that the total probabilistic rate of entry into new states cannot exceed the rate of departure from the old state, and one might have expected equality here. Kolmogorov (**10**) has shown by a counter-example that this expectation is unjustified, and he has also shown that q_{ij} can be $-\infty$ if $j = i$. [For a further study of these examples, see (**9**).] In view of the restrictions on the algebraic signs, (3) would fail in the worst possible way if

$$q_{00} = -\infty, \qquad q_{0j} = 0 \quad (\text{all } j \neq 0). \tag{4}$$

Kolmogorov has remarked in a personal communication at the Amsterdam Congress, 1954, that an (honest) process having (4) as a row of its q-matrix could be constructed by the methods of his paper (**10**). Such a process would be very remarkable; one could say of the system that 'it leaves state 0 with infinite velocity and goes nowhere in particular'. The associated semigroup would be very interesting, but unfortunately the details of Kolmogorov's construction are not available. In § 5 of the present paper I shall use the Hille–Yosida theorem to construct a transition semigroup such that the associated process has (4) as a row of its q-matrix.

In statistical practice it has been usual to postulate a set of q's satisfying the restrictive conditions,

(*a*) q_{ij} is finite for all i and j;

(*b*) $\sum_{\alpha=0}^{\infty} q_{i\alpha} = 0$ for each i;

and then to expect that the p-functions of an associated (honest) process will be uniquely determined by the so-called *backward* and *forward* differential equations:

$$\frac{d}{dt} p_{ij}(t) = \sum_{\alpha=0}^{\infty} q_{i\alpha} p_{\alpha j}(t), \tag{5 b}$$

$$\frac{d}{dt} p_{ij}(t) = \sum_{\alpha=0}^{\infty} p_{i\alpha}(t) q_{\alpha j}. \tag{5 f}$$

It has been known for some time [Doob (**1**)] that the q-matrix does *not* characterize a process uniquely, even though we insist on its honesty and even if (*a*) and (*b*) are satisfied. It is also known that for some q-matrices satisfying (*a*) and (*b*) there is *no* (honest) process both

† It is convenient to use the adjective 'honest' to emphasize that (II) holds; similarly 'dishonest' will mean that (II*) holds *with inequality* for some i and t.

admitting these q's and satisfying (5b) and (5f) [see, for example, the discussion of birth processes in Feller (**3**)]. We may therefore ask: *suppose that there does exist an (honest) process having a given set of q's (satisfying (a) and (b)) and such that the p-functions satisfy both* (5b) *and* (5f); *is this process then necessarily unique?*

I shall (in § 4) present an example to show that it is *not*. This example of non-unicity is stronger than that given by Ledermann and Reuter (**11**) in that one of their two solutions was a quasi-process.

The implications of this result in semigroup theory are best seen by considering it in relation to the following two theorems. Let us call a q-matrix *conservative* when (a) and (b) are satisfied, and when this is so let us define

$$(Qx)_j \equiv \sum_{\alpha=0}^{\infty} x_\alpha q_{\alpha j} \tag{6}$$

for all x in $\mathscr{D}(Q)$, this domain being defined to be that part of l in which

(i) the series (6) is absolutely convergent for each j,

(ii) $$\sum_{\beta=0}^{\infty} |(Qx)_\beta| < \infty.$$

All the 'terminating' vectors (vectors $x \in l$ such that $x_\alpha = 0$ for all sufficiently large α) will lie in $\mathscr{D}(Q)$, and we shall write Q_0 for the contraction of Q to the set $\mathscr{D}_0$ of all terminating vectors as domain. We then have

THEOREM A (Kendall and Reuter). *An (honest) process will have a conservative q-matrix if and only if for the associated transition semigroup all the terminating vectors lie in $\mathscr{D}(\Omega)$. When this is so, then Ω is an extension of Q_0,*

$$q_{ij} = (\Omega u^i)_j,$$

the p-functions have continuous derivatives and the BACKWARD *equations* (5b) *hold for $t \geqslant 0$.*

THEOREM B (Reuter). *If a process (transition semigroup) satisfies the conditions of Theorem A, then the* FORWARD *equations* (5f) *will hold for $t \geqslant 0$ if and only if Ω is a contraction of Q.*

When these results are taken into account, the uniqueness problem can be posed as follows: *given a conservative set of q's, is it possible for the infinitesimal generator of more than one* TRANSITION *semigroup to satisfy the operator-inequality*

$$Q_0 \subseteq \Omega \subseteq Q? \tag{7}$$

It will be shown in § 4 that the set of solutions Ω to (7) can have the power of the continuum. In what follows we shall not use the full force of Theorems A and B, and their proofs will not be given here. We only need the more elementary fact that (5b) and (5f) always hold if

$$Q_0 \subseteq \Omega \subseteq Q;$$

for a proof of this, see (**9**).

It will be evident how much this work owes to Lévy's analysis (**12, 13**) of Markovian pathologies. I wish also to thank Professor A. N. Kolmogorov, Mr. J. M. Hammersley, and Mr. G. E. H. Reuter for helpful discussions of these problems; to Mr. Reuter I am further indebted for permission to quote Theorems A and B and for a large number of detailed suggestions which will not be acknowledged individually.

2. The Hille–Yosida theorem for an abstract L-space

For the general theory of semigroups reference should be made to Hille (**4, 5**) and Yosida (**16, 17**). Semigroups will be constructed in §§ 3–5 with the aid of the Hille–Yosida theorem in the following modified form which proves very convenient when (as here) the underlying space is an abstract L-space satisfying the axioms I to IX of Kakutani (**7**).

Theorem C (Hille–Yosida theorem). *Let X be an abstract L-space in the sense of Kakutani, let $\mathscr{D}(\Omega)$ be a linear subset of X and let Ω map $\mathscr{D}(\Omega)$ linearly into X. Then Ω on $\mathscr{D}(\Omega)$ will be the infinitesimal generator of a contraction (transition) semigroup of operators on X if and only if the following three conditions are satisfied:*

(1) *$\mathscr{D}(\Omega)$ is dense in X;*

(2) *$(e, \Omega x) \leqslant 0$ $(= 0)$ whenever $0 \leqslant x \in \mathscr{D}(\Omega)$;*

(3) *whenever $\lambda > 0$ and $0 \leqslant x \in X$,*

then the equation $$\lambda\xi - \Omega\xi = x \quad (\xi \in \mathscr{D}(\Omega)) \tag{8}$$

has a unique solution ξ, and this $\xi \geqslant 0$.

Here e is the bounded positive linear functional defined by

$$(e, x) = \|x_+\| - \|x_-\|.$$

The necessity of (1) to (3) follows from the introductory lemmas in Reuter's paper (**14**). Their sufficiency (which is what is needed here) is proved as follows. Let $\xi \equiv J_\lambda x$ be the unique solution to (8) when $\lambda > 0$ and $0 \leqslant x \in X$; we are given that $\xi \geqslant 0$, and it is clear that $J_\lambda 0 = 0$. If now x is *any* element of X, then we can consistently define

$$J_\lambda x \equiv J_\lambda(x_+) - J_\lambda(x_-).$$

We shall have $J_\lambda x \in \mathscr{D}(\Omega)$ and

$$(\lambda I - \Omega)J_\lambda x = x_+ - x_- = x.$$

If ξ is an arbitrary solution to (8) for any given $x \in X$ and if $\eta \equiv \xi - J_\lambda x$, then we shall have $\eta \in \mathscr{D}(\Omega)$ and $\lambda\eta - \Omega\eta = 0$, so that $\eta = J_\lambda 0 = 0$. Thus (8) has a unique solution for *all* $x \in X$, and it satisfies $\xi \geqslant 0$ whenever $x \geqslant 0$. Also it is now evident that J_λ is linear over X.

From (2) we now have that $(e, \Omega J_\lambda x) \leqslant 0$ $(= 0)$ whenever $x \geqslant 0$, and so (for positive x)

$$\|\lambda J_\lambda x\| = (e, \lambda J_\lambda x) = (e, x) + (e, \Omega J_\lambda x) \leqslant (e, x) = \|x\|$$

(with equality if the equality sign holds in (2)). It now easily follows that

$$\|\lambda J_\lambda x\| = \|\lambda J_\lambda(x_+) - \lambda J_\lambda(x_-)\| \leqslant \|x_+\| + \|x_-\| = \|x\|$$

for all $x \in X$, and that $\|\lambda J_\lambda x\| = \|x\|$ if *both* $x \geqslant 0$ and the equality sign holds in (2). The required result now follows at once from the Hille–Yosida theorem in its usual form.

In our applications X will be l, and e will be the functional defined by

$$(e, x) = \sum_{\alpha=0}^{\infty} x_\alpha.$$

In § 4 we shall have occasion to use the fact that *the following sets all have the power c of the continuum:*

(*a*) all permutations f of $(..., -1, 0, 1, ...)$ for which 0 is a fixpoint;
(*b*) all $x \in l$;
(*c*) all transition operators on l;
(*d*) all transition semigroups of operators on l;
(*e*) all temporally homogeneous Markov processes having a countable infinity of states.

Proof (*a*) It will clearly be sufficient to consider the set of permutations g of the positive integers. Each permutation g is a mapping from Z (the set of integers) into itself, distinct permutations determining distinct mappings, and the cardinal number of such mappings is c. Also, given an irrational binary 'decimal', let the kth unit occur in the m_kth place and the kth zero in the n_kth place; then

$$g \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & . & . & . \\ m_1 & n_1 & m_2 & n_2 & . & . & . \end{pmatrix}$$

is a permutation of Z and distinct 'decimals' will determine distinct permutations. Thus there are exactly c permutations.

(*b*) The space l is a subset of the set of all mappings from Z into the real line, and it contains as a subset all terminating vectors. Thus l has cardinal c.

(*c*) A transition operator on l can be identified with the sequence of rows in its matrix representation (1), each of which is a positive l-vector of unit norm. It is now easy to show that the cardinal number is again equal to c, on using the preceding result.

(*d*) If A is a transition operator, then

$$P_t \equiv \exp(A-I)t \quad (t \geqslant 0)$$

defines a transition semigroup with $\Omega = A-I$. This gives a (1, 1) mapping from set (*c*) into set (*d*). If $\{P_t : t \geqslant 0\}$ is a transition semigroup then

$$Ax \equiv J_1 x \equiv \int\limits_0^\infty e^{-t} P_t x \, dt \quad (x \in l)$$

defines a transition operator A such that the range of A is $\mathscr{D}(\Omega)$ and the inverse of A is $I-\Omega$. Thus we also have a (1, 1) mapping from set (*d*) into set (*c*), and so (*d*) has the power of the continuum.

(*e*) This follows immediately from the preceding result.

3. First example: The 'flash'

We begin by constructing the transition semigroup associated with a process introduced by Lévy [(**12**), 366, example 3°]; this will then play a fundamental role in the later constructions. *Let*

$$a_n > 0 \quad (n = ..., -1, 0, 1, ...) \quad \textit{and let} \quad \sum_{-\infty}^{\infty} 1/a_\alpha < \infty.$$

We shall now abandon the use of the non-negative integers as an index-set, and we shall take as a typical element of l the vector

$$x \equiv (..., x_{-1}, x_0, x_1, ...)$$

with

$$\|x\| \equiv \sum_{-\infty}^{\infty} |x_\alpha| < \infty.$$

We introduce a conservative q-matrix with elements given by

$$q_{ij} = \begin{cases} -a_i & (j = i), \\ +a_i & (j = i+1), \\ 0 & \textit{otherwise}, \end{cases}$$

and we note that a vector x lies in $\mathscr{D}(Q)$ if and only if (i) $x \in l$ and (ii) $\sum |a_{\alpha-1} x_{\alpha-1} - a_\alpha x_\alpha| < \infty$. When $x \in \mathscr{D}(Q)$, then

$$(Qx)_j \equiv a_{j-1} x_{j-1} - a_j x_j.$$

The condition (ii) implies the existence of the limits

$$Ux \equiv \lim_{\alpha\to+\infty} a_\alpha x_\alpha, \qquad Lx \equiv \lim_{\alpha\to-\infty} a_\alpha x_\alpha,$$

and so (because $\sum 1/a_\alpha < \infty$) (ii) implies (i). Finally, for all $x \in \mathscr{D}(Q)$,

$$(e, Qx) = Lx - Ux.$$

We now define the operator Ω to be the contraction of Q from $\mathscr{D}(Q)$ to the smaller domain $\mathscr{D}(\Omega)$ determined by the side-condition,

$$Lx = Ux. \tag{9}$$

It is easily verified that Ω maps the linear set $\mathscr{D}(\Omega)$ linearly into l; also each unit coordinate vector is in $\mathscr{D}(Q)$ and trivially satisfies (9), so that $\mathscr{D}_0 \subseteq \mathscr{D}(\Omega)$ and $Q_0 \subseteq \Omega \subseteq Q$. One can now immediately check that conditions (1) and (2) (*with equality*) of Theorem C are satisfied; we therefore proceed to consider condition (3).

Let $\lambda > 0$ and $0 \leqslant x \in l$; then the vector ξ will satisfy (8) if and only if

$$(\lambda + a_j)\xi_j - a_{j-1}\xi_{j-1} = x_j \quad (\text{all } j), \tag{10}$$

$$L\xi = U\xi, \tag{11}$$

$$\xi \in l. \tag{12}$$

There is no need to add the requirement that

$$\sum |a_{\alpha-1}\xi_{\alpha-1} - a_\alpha \xi_\alpha| < \infty,$$

because this follows from (10) and (12). Now, if we write $c_n \equiv 1 + \lambda/a_n$, then (10) is equivalent to

$$a_n \xi_n = \left(\frac{x_n}{c_n} + \frac{x_{n-1}}{c_n c_{n-1}} + \frac{x_{n-2}}{c_n c_{n-1} c_{n-2}} + \dots\right) + \mu \prod_{\alpha=-\infty}^{n} \frac{1}{c_\alpha}, \tag{10*}$$

the series and the product both being convergent. From (10*) we find that $L\xi = \mu$ and that

$$U\xi = \sum_{\alpha=-\infty}^{\infty} x_\alpha \prod_{\beta=\alpha}^{\infty} \frac{1}{c_\beta} + \mu \prod_{\beta=-\infty}^{\infty} \frac{1}{c_\beta},$$

so that (10) and (11) together are equivalent to (10*) and

$$\mu = \sum_{\alpha=-\infty}^{\infty} x_\alpha \prod_{\beta=\alpha}^{\infty} \frac{1}{c_\beta} \left\{1 - \prod_{\gamma=-\infty}^{\infty} \frac{1}{c_\gamma}\right\}^{-1}; \tag{11*}$$

we note that μ is finite and non-negative because $0 < \prod c_j^{-1} < 1$. It will now follow from Theorem C that Ω generates a transition semigroup

if it can be shown that (12) is satisfied when ξ is defined by (10*) and (11*). This is easily shown to be so by noting that

$$a_n|\xi_n| \leqslant \|x\|+\mu$$

and recalling that $\sum 1/a_\alpha < \infty$.

We shall call this the 'flash' semigroup (or process) with parameters (a_n: $n = ..., -1, 0, 1,...$). Because $Q_0 \subseteq \Omega \subseteq Q$, the p-functions must satisfy the backward and forward equations,

$$p'_{ij}(t) = a_i(p_{i+1,j}-p_{ij}),$$

$$p'_{ij}(t) = a_{j-1}p_{i,j-1}-a_j p_{ij}.$$

It is a simple matter to work out the Laplace transforms of the p-functions and so to identify the process with that described by Lévy. For example, we find that

$$\int\limits_0^\infty e^{-\lambda t}p_{00}(t)\,dt = \frac{1}{\lambda+a_0}+\sum_{r=0}^\infty\Big(\prod_0^\infty \frac{a_\alpha}{\lambda+a_\alpha}\Big)\Big(\prod_{-\infty}^\infty \frac{a_\beta}{\lambda+a_\beta}\Big)^r\Big(\prod_{-\infty}^{-1} \frac{a_\gamma}{\lambda+a_\gamma}\Big)\frac{1}{\lambda+a_0}.$$

Informally we may say that the system 'flashes' through the states in the order of the labelling, spending a mean time $1/a_n$ in the nth state; each 'flash' has the finite mean duration $\sum 1/a_\alpha$. When a flash is completed, the system moves instantaneously from $+\infty$ to $-\infty$ and a new flash is commenced.

4. Second example: The 'flash of flashes'

We now take as a typical l-vector the double sequence

$$x \equiv (..., x^{-1}, x^0, x^1,...),$$

where $$x^s \equiv (..., x^s_{-1}, x^s_0, x^s_1,...)$$

and $$\|x\| \equiv \sum_{-\infty}^\infty \sum_{-\infty}^\infty |x^\sigma_\alpha| < \infty.$$

Let $a^s_n > 0$ $(s, n = ..., -1, 0, 1,...)$, *let* $\sum\sum 1/a^\sigma_\alpha < \infty$ *and let* f *be any permutation of* $(..., -1, 0, 1,...)$ *for which* 0 *is a fixpoint. We introduce a conservative* q-*matrix with elements given by*

$$q^{rs}_{mn} = 0 \quad (r \neq s),$$

$$q^{ss}_{mn} = \begin{cases} -a^s_m & (n = m), \\ +a^s_m & (n = m+1), \\ 0 & \textit{otherwise}. \end{cases}$$

Then $x \in \mathscr{D}(Q)$ if and only if (i) $x \in l$ and (ii) $\sum\sum |a^{\sigma}_{\alpha-1} x^{\sigma}_{\alpha-1} - a^{\sigma}_{\alpha} x^{\sigma}_{\alpha}| < \infty$, and then

$$(Qx)^s_n \equiv a^s_{n-1} x^s_{n-1} - a^s_n x^s_n.$$

The condition (ii) implies the existence of the limits

$$U^s x \equiv \lim_{\alpha \to +\infty} a^s_{\alpha} x^s_{\alpha}, \qquad L^s x \equiv \lim_{\alpha \to -\infty} a^s_{\alpha} x^s_{\alpha}.$$

We now define the operator Ω^f to be the contraction of Q from $\mathscr{D}(Q)$ to the smaller domain $\mathscr{D}(\Omega^f)$ determined by the side-conditions

$$U^{f(s)} x = L^{f(s+1)} x \quad (s = ..., -1, 0, 1, ...), \tag{13_f}$$

and

$$\lim_{\sigma \to +\infty} U^{f(\sigma)} x = \lim_{\sigma \to -\infty} L^{f(\sigma)} x. \tag{14_f}$$

Notice that, when $x \in \mathscr{D}(Q)$, then

$$\sum_{-\infty}^{\infty} |L^{f(\sigma)} x - U^{f(\sigma)} x| < \infty.$$

If (13_f) holds, it will then follow that

$$\sum_{-\infty} |L^{f(\sigma)} x - L^{f(\sigma+1)} x| \quad \text{and} \quad \sum^{\infty} |U^{f(\sigma-1)} x - U^{f(\sigma)} x|$$

are both convergent and consequently that the limits in (14_f) exist. As before, Ω^f maps the linear set $\mathscr{D}(\Omega^f)$ linearly into l, and the unit coordinate vectors all lie in $\mathscr{D}(\Omega^f)$, so that

$$\mathscr{D}_0 \subseteq \mathscr{D}(\Omega^f), \qquad Q_0 \subseteq \Omega^f \subseteq Q.$$

Conditions (1) and (2) (*with equality*) of Theorem C are satisfied because $\mathscr{D}_0$ is dense and because, when $x \in \mathscr{D}(\Omega^f)$,

$$\begin{aligned}(e, \Omega^f x) = \sum\sum (\Omega^f x)^{\sigma}_{\alpha} &= \sum\sum (Qx)^{\sigma}_{\alpha} \\ &= \sum \{L^{\sigma} x - U^{\sigma} x\} \\ &= \sum \{L^{f(\sigma)} x - U^{f(\sigma)} x\} \\ &= \sum \{L^{f(\sigma)} x - L^{f(\sigma+1)} x\} \\ &= \lim_{S \to +\infty} \{L^{f(-S)} x - U^{f(S)} x\} = 0.\end{aligned}$$

It remains for us to verify condition (3).

Let $\lambda > 0$ and $0 \leqslant x \in l$; then the vector ξ will satisfy (8) if and only if

$$(\lambda + a^s_n) \xi^s_n - a^s_{n-1} \xi^s_{n-1} = x^s_n \quad (\text{all } s, n), \tag{15}$$

$$U^{f(s)} \xi = L^{f(s+1)} \xi \quad (\text{all } s), \tag{16}$$

$$\lim_{\sigma \to +\infty} U^{f(\sigma)} \xi = \lim_{\sigma \to -\infty} L^{f(\sigma)} \xi, \tag{17}$$

and

$$\xi \in l. \tag{18}$$

As in § 3, we can drop the condition (ii) for $\xi \in \mathscr{D}(Q)$ in virtue of (15). Now (15) is equivalent to

$$a_n^s \xi_n^s = \left(\frac{x_n^s}{c_n^s} + \frac{x_{n-1}^s}{c_n^s c_{n-1}^s} + \dots\right) + \mu^s \prod_{\alpha=-\infty}^{n} \frac{1}{c_\alpha^s}, \tag{15*}$$

where $c_n^s \equiv 1 + \lambda/a_n^s$, the series and product both being convergent, and we then find (just as in § 3) that $L^s\xi = \mu^s$ and that

$$U^s\xi = \sum_{\alpha=-\infty}^{\infty} x_\alpha^s \prod_{\beta=\alpha}^{\infty} \frac{1}{c_\beta^s} + \mu^s \prod_{\beta=-\infty}^{\infty} \frac{1}{c_\beta^s}.$$

Thus (15) and (16) together are equivalent to (15*) and

$$\mu^{f(s+1)} = \mu^{f(s)} d^{f(s)} + y^{f(s)} \quad (\text{all } s), \tag{16*}$$

where $$d^s \equiv \prod_{\beta=-\infty}^{\infty} \frac{1}{c_\beta^s}, \qquad y^s \equiv \sum_{\alpha=-\infty}^{\infty} x_\alpha^s \prod_{\beta=\alpha}^{\infty} \frac{1}{c_\beta^s}.$$

Now (16*) in its turn is equivalent to

$$\mu^{f(s)} = (y^{f(s-1)} + y^{f(s-2)} d^{f(s-1)} + \dots) + \nu \prod_{\sigma=-\infty}^{s-1} d^{f(\sigma)}, \tag{16**}$$

and (17), in the presence of (15*) and (16**), is equivalent to

$$\nu = \sum_{\sigma=-\infty}^{\infty} y^{f(\sigma)} \prod_{\tau=\sigma+1}^{\infty} d^{f(\tau)} \left\{1 - \prod_{\rho=-\infty}^{\infty} d^{f(\rho)}\right\}^{-1}. \tag{17*}$$

It is to be noticed that ν is non-negative and finite because

$$0 < \prod_{-\infty}^{\infty} d^{f(\rho)} = \prod_{-\infty}^{\infty} \prod_{-\infty}^{\infty} 1/c_\alpha^\sigma < \infty.$$

We have now only to verify that ξ as defined by (15*), (16**), and (17*) satisfies (18). From (15*),

$$a_n^s |\xi_n^s| \leqslant \|x\| + \sup_{-\infty < s < \infty} |\mu^s|,$$

and the right-hand side is finite because

$$\mu^{f(s)} \to \nu \quad \text{when } s \to \pm\infty.$$

Thus $\xi \in l$ because $\sum\sum 1/a_\alpha^\sigma < \infty$, and so Ω^f generates a transition semigroup such that

$$Q_0 \subseteq \Omega^f \subseteq Q.$$

We shall accordingly have a continuous infinity of transition semigroups whose infinitesimal generators can be interpolated between Q_0 and Q (so that the associated (honest) processes all have the given conservative q-matrix and satisfy both the backward and forward equations) if we can show that $\Omega^f \neq \Omega^g$ unless $f = g$.

Let Ω denote Ω^f when $f(s) \equiv s$. Then because of the group-property of permutations it will suffice to show that

$$\Omega^f = \Omega \quad \text{implies} \quad f(s) \equiv s. \tag{19}$$

Proof. Put

$$l^s \equiv \begin{cases} 1/s & \text{when } s \geqslant 1, \\ 0 & \text{when } s = 0, \\ -1/s & \text{when } s \leqslant -1, \end{cases}$$

and $u^s \equiv l^{s+1}$, and consider the vector x having the components

$$x_n^s \equiv \begin{cases} u^s/a_n^s & \text{when } n \geqslant 1, \\ l^s/a_n^s & \text{when } n \leqslant 0. \end{cases}$$

Then

$$x \in l, \qquad x \in \mathscr{D}(Q), \qquad L^s x = l^s$$

and $U^s x = u^s$, and consequently $x \in \mathscr{D}(\Omega)$. Suppose that x also lies in $\mathscr{D}(\Omega^f)$. Then we must have $l^{f(s+1)} = u^{f(s)} = l^{f(s)+1}$; but the l's are all different and so $f(s+1) = f(s)+1$, $f(0) = 0$. Thus $f(s) \equiv s$.

The stochastic motion described by Ω^f can be determined from the above formulae by the method indicated at the end of § 3. First, if $f(s) \equiv s$ and if the system is initially in the state $\binom{s}{n}$, it completes the flash with parameters $(..., a_{-1}^s, a_0^s, a_1^s,...)$ and then at once commences the $(s+1)$th flash, and so on. With probability one the total time required to pass once through every flash will be finite, with mean value $\sum\sum 1/a_\alpha^\sigma$, and, when the system has run through all the flashes subsequent to the initial one, it instantaneously starts moving through the flash-of-flashes from 'the beginning', $\binom{-\infty}{-\infty}$, this cycle being repeated indefinitely. The motion is difficult to describe in a few words but is quite easy to visualize if one thinks of the motion of the flying spot on an (infinite) television screen. This is the motion described by Ω. In the motion described by Ω^f the flashes are followed in the order assigned by the permutation f, instead of in their natural order. It is to be noticed that we had to require $f(0) = 0$ in order to eliminate shift-permutations; these generate essentially the same stochastic motion as Ω.

The remarks at the end of § 2 concerning the cardinality of various infinite sets make it clear that we have here a set of backward and forward equations (whose coefficients constitute a conservative q-matrix) which is satisfied by the p-functions of precisely as many (honest) processes as there are such processes altogether. Thus the non-unicity might be described as maximal.†

† [*Note added in proof.* Professor Feller informs me that in a forthcoming paper he will present an exhaustive analysis of such non-unicity phenomena.]

5. Third example: A sequence of flashes communicating via an instantaneous state

In (9) G. E. H. Reuter and I constructed the transition semigroup associated with Kolmogorov's process K1 which has

$$(-\infty, 1, 1, 1, \ldots)$$

as a row of its q-matrix, the corresponding state being an instantaneous one in the terminology of Lévy. At the Amsterdam Congress of 1954 Professor Kolmogorov remarked to us that one could make up a process having

$$(-\infty, 0, 0, 0, \ldots) \tag{20}$$

as a row of its q-matrix; this would represent the worst possible example of inequality in the semi-conservation condition (3). The details of Kolmogorov's construction are not available, but I shall now show how one can construct such a process by direct semigroup methods. The key idea is the replacement of each stable state in K1 by a flash of suitable mean duration.

This time we take as a typical l-vector the double sequence,

$$x \equiv (x^0, x^1, x^2, \ldots),$$

where x^0 is a real number,

$$\bar{x}^s \equiv (\ldots, x^s_{-1}, x^s_0, x^s_1, \ldots) \quad (s \geqslant 1),$$

and

$$\|x\| \equiv |x^0| + \sum_{\sigma=1}^{\infty} \sum_{\alpha=-\infty}^{\infty} |x^\sigma_\alpha| < \infty.$$

The state $\binom{0}{\cdot}$ will be instantaneous, and the stable states $\binom{s}{n}$ for a given $s \geqslant 1$ and $n = \ldots, -1, 0, 1, \ldots$ will constitute the sth flash. Because of (20) the q-matrix will no longer be conservative, so that we must not expect to be able to define Ω as a contraction of Q.

Let $a^s_n > 0$ ($s = 1, 2, 3, \ldots$; $n = \ldots, -1, 0, 1, \ldots$) and let $\sum\sum 1/a^\sigma_\alpha < \infty$. We define $\mathscr{D}(\Omega)$ to be the set of vectors x such that

(i) $$\sum_{\sigma=1}^{\infty} \sum_{\alpha=-\infty}^{\infty} |a^\sigma_{\alpha-1} x^\sigma_{\alpha-1} - a^\sigma_\alpha x^\sigma_\alpha| \equiv C(x) < \infty$$

and

(ii) $$L^s x \equiv \lim_{\alpha \to -\infty} a^s_\alpha x^s_\alpha = x^0 \quad (\textit{all } s \geqslant 1).$$

As in §§ 3 and 4, $L^s x$ and $U^s x$ exist whenever (i) holds, so that (ii) is meaningful when associated with (i). It also follows from (i) that

$$\sum_{\sigma=1}^{S} |L^\sigma x - a^\sigma_n x^\sigma_n| \leqslant C(x) < \infty$$

for each n and S, and so, if $x \in \mathscr{D}(\Omega)$,

$$\sum (L^\sigma x - U^\sigma x) \equiv \sum (x^0 - U^\sigma x)$$

is absolutely convergent and

$$a_n^s |x_n^s| \leqslant |x^0| + C(x);$$

this last inequality shows that $\mathscr{D}(\Omega) \subseteq l$. *We now define Ωx for $x \in \mathscr{D}(\Omega)$ as follows*:

$$\left.\begin{aligned} (\Omega x)^0 &\equiv \sum_{\sigma=1}^{\infty} (U^\sigma x - x^0), \\ (\Omega x)_n^s &\equiv a_{n-1}^s x_{n-1}^s - a_n^s x_n^s \quad (s \geqslant 1, \textit{ all } n) \end{aligned}\right\}. \tag{21}$$

It is clear that $\mathscr{D}(\Omega)$ is linear and that Ω maps $\mathscr{D}(\Omega)$ linearly into l. All the unit coordinate vectors u_n^s lie in $\mathscr{D}(\Omega)$, but u^0 does not. However, $\mathscr{D}(\Omega)$ contains the vector x for which

$$x^0 \equiv 1, \qquad x_n^s \equiv 1/a_n^s \quad (s \geqslant 1, \text{ all } n),$$

and so $\mathscr{D}(\Omega)$ is dense. The second condition in Theorem C is also satisfied (*with equality*) because, if $x \in \mathscr{D}(\Omega)$,

$$\begin{aligned} (e, \Omega x) &= \sum_{\sigma=1}^{\infty} (U^\sigma x - x^0) + \sum_{\sigma=1}^{\infty} (L^\sigma x - U^\sigma x) \\ &= \sum_{\sigma=1}^{\infty} (L^\sigma x - x^0) = 0. \end{aligned}$$

We therefore turn to condition (3).

Let $\lambda > 0$ and $0 \leqslant x \in l$. Then (8) is equivalent to

$$\lambda \xi^0 + \sum_{\sigma=1}^{\infty} (\xi^0 - U^\sigma \xi) = x^0, \tag{22}$$

$$(\lambda + a_n^s)\xi_n^s - a_{n-1}^s \xi_{n-1}^s = x_n^s \quad (s \geqslant 1, \text{ all } n), \tag{23}$$

$$L^s \xi = \xi^0 \quad (s \geqslant 1), \qquad \xi \in l. \tag{24, 25}$$

As usual (25) replaces (i) in virtue of (23). Putting

$$c_n^s \equiv 1 + \lambda / a_n^s$$

we find that (23) is equivalent to

$$a_n^s \xi_n^s = \left(\frac{x_n^s}{c_n^s} + \frac{x_{n-1}^s}{c_n^s c_{n-1}^s} + \ldots \right) + \mu^s \prod_{\alpha=-\infty}^{n} \frac{1}{c_\alpha^s}, \tag{23*}$$

both series and product being convergent, and (23*) implies that $L^s \xi = \mu^s$ and that

$$U^s \xi = \sum_{\alpha=-\infty}^{\infty} x_\alpha^s \prod_{\beta=\alpha}^{\infty} \frac{1}{c_\beta^s} + \mu^s \prod_{\beta=-\infty}^{\infty} \frac{1}{c_\beta^s}.$$

Thus (23) and (24) are together equivalent to (23*) and

$$\mu^s = \xi^0 \quad (s \geqslant 1), \tag{24*}$$

and (22–24) are together equivalent to (23*), (24*) and

$$\xi^0 = \left\{x^0 + \sum_{\sigma=1}^{\infty} \sum_{\alpha=-\infty}^{\infty} x_\alpha^\sigma \prod_{\beta=\alpha}^{\infty} \frac{1}{c_\beta^\sigma}\right\}\left\{\lambda + \sum_{\rho=1}^{\infty}\left[1 - \prod_{\gamma=-\infty}^{\infty} \frac{1}{c_\gamma^\rho}\right]\right\}^{-1}. \tag{22*}$$

We have here made use of the inequalities

$$1/c_n^s > e^{-\lambda/a_n^s},$$

$$\prod_{\gamma=-\infty}^{\infty} \frac{1}{c_\gamma^s} > \exp\left[-\lambda \sum_{\gamma=-\infty}^{\infty} \frac{1}{a_\gamma^s}\right]$$

$$> 1 - \lambda \sum_{\gamma=-\infty}^{\infty} \frac{1}{a_\gamma^s},$$

and
$$0 < \sum_{\sigma=1}^{\infty}\left[1 - \prod_{\gamma=-\infty}^{\infty} \frac{1}{c_\gamma^\sigma}\right] < \lambda \sum\sum \frac{1}{a_\alpha^\sigma} < \infty.$$

It will be observed that the (unique) ξ defined by (22*)–(24*) is positive or zero, and we have only to show that it satisfies (25). But from (23*) and (24*) we have

$$a_n^s |\xi_n^s| \leqslant \|x\| + |\xi^0|,$$

and so $\xi \in l$. Thus Ω on $\mathscr{D}(\Omega)$ generates a transition semigroup, and we now have to find the $\binom{0}{}$th row of its q-matrix.

For this purpose we need the following lemma, valid for any transition or contraction semigroup of operators on l.

LEMMA. $$q_{ij} = \lim_{\lambda\to\infty} (v^j, \lambda(\lambda J_\lambda - I)u^i).$$

Here we have temporarily reverted to the set of integers as state-labels, v^j is the jth unit coordinate vector in $l^* \equiv m$ and J_λ is the resolvent operator defined by

$$J_\lambda x = \int_0^\infty e^{-\lambda t} P_t x \, dt \quad (\lambda > 0)$$

or by
$$(\lambda I - \Omega) J_\lambda x = x \quad (\lambda > 0).$$

Proof of the lemma. We have

$$(v^j, \lambda(\lambda J_\lambda - I)u^i) = \int_0^\infty t^{-1}[p_{ij}(t) - \delta_{ij}]\lambda t e^{-\lambda t}\lambda \, dt,$$

and the result is now immediate if $i \neq j$, or if both $i = j$ and q_{ii} is finite.

If $i = j$ and if $q_{ii} = -\infty$, then we first choose a positive Δ_N such that

$$1-p_{ii}(t) \geqslant Nt \quad \text{when } 0 \leqslant t \leqslant \Delta_N,$$

and we note that
$$(v^j, \lambda(\lambda J_\lambda - I)u^i) = \int_0^\Delta + \int_\Delta^\infty,$$

where
$$\int_0^\Delta \leqslant -N \int_0^{\lambda\Delta} we^{-w}\,dw$$

and
$$\left|\int_\Delta^\infty\right| \leqslant \lambda e^{-\lambda\Delta}.$$

The required result now follows on letting first λ and then N tend to infinity.

We now use the lemma to evaluate q^{00} and $q^{0s}_{\cdot n}$ $(s \geqslant 1)$. First,

$$(v^0, \lambda(\lambda J_\lambda - I)u^0) = \frac{-\lambda \sum [1 - \prod 1/c^\sigma_\alpha]}{\lambda + \sum\limits_\sigma \left[1 - \prod\limits_\alpha 1/c^\sigma_\alpha\right]},$$

and $F^s(\lambda) \equiv \prod\limits_\alpha 1/c^s_\alpha \downarrow 0$ as $\lambda \to \infty$, so that $q^{00} = -\infty$. Next,

$$0 \leqslant (v^s_n, \lambda(\lambda J_\lambda - I)u^0) = \frac{\lambda^2}{a^s_n} \prod_{\alpha=-\infty}^{n} \frac{1}{c^s_\alpha} \left[\lambda + \sum_{\sigma=1}^{\infty} [1 - F^\sigma(\lambda)]\right]^{-1}$$
$$< \prod_{\alpha=-\infty}^{n-1} 1/c^s_\alpha$$
$$< a^s_{n-1}/(\lambda + a^s_{n-1}) \to 0$$

as $\lambda \to \infty$, so that $q^{0s}_{\cdot n} = 0$. This completes the proof that the $\binom{0}{\cdot}$th row of the q-matrix is as shown at (20) above.

A complete analysis of the stochastic motion would be very tedious and has not been carried out. The reader who is interested should find all the necessary apparatus in § 7 of (9); he has only to replace each stable state by a flash with the indicated parameters, and this will be left to him as a somewhat painful exercise. The arguments illustrated at the end of § 3 of the present paper, when applied to show how the system moves from $\binom{r}{m}$ to $\binom{s}{n}$ (where $r \neq s$ and $r, s \geqslant 1$) merely indicate that it begins by completing the rth flash and then enters the instantaneous state $\binom{0}{\cdot}$. At a later epoch it leaves $\binom{0}{\cdot}$ to climb up the sth flash as far as the state $\binom{s}{n}$. A description of the motion between the first entry into and the last exit from the instantaneous state $\binom{0}{\cdot}$ cannot be given in such simple terms and will not be dealt with further here.

It will, however, be useful to obtain the integral equation which determines $p^{00}(t)$. The corresponding integral equation for K1 [for this see (**10**) and (**9**)] proved to be of independent interest and has been investigated in detail by Reuter (**15**).

The Laplace transform $\Phi(\lambda)$ of $\phi(t) \equiv p^{00}(t)$ is given as usual by (2); in this way we find that

$$\Phi(\lambda)\{1+K(\lambda)\} = 1/\lambda \quad (\lambda > 0), \tag{26}$$

where

$$K(\lambda) \equiv \sum_{\sigma=1}^{\infty} \frac{1}{\lambda}\left\{1 - \prod_{\alpha=-\infty}^{\infty} \frac{a_\alpha^\sigma}{\lambda+a_\alpha^\sigma}\right\}. \tag{27}$$

The following technical device is adopted to avoid the quoting of non-trivial theorems about infinite convolutions; it is *not* directly connected with a representation of the stochastic motion. Let v_n^s ($s \geqslant 1$, all n) be independent random variables distributed thus:

$$e^{-a_n^s v_n^s} a_n^s \, dv_n^s \quad (0 < v_n^s < \infty);$$

if we put

$$V^s \equiv \sum_{\alpha=-\infty}^{\infty} v_\alpha^s,$$

then with probability one all the V's will be positive and finite, and

$$\mathbf{E}(e^{-\lambda V^s}) = \int_{0-}^{\infty} e^{-\lambda V}\, dH^s(V) = \prod_{\alpha=-\infty}^{\infty} \frac{a_\alpha^s}{\lambda+a_\alpha^s},$$

where $H^s(.)$ is the distribution function of V^s. Thus

$$\frac{1}{\lambda}\left\{1 - \prod_{\alpha=-\infty}^{\infty} \frac{a_\alpha^s}{\lambda+a_\alpha^s}\right\} = \int_0^{\infty} [1-H^s(V)]e^{-\lambda V}\, dV,$$

and $K(\lambda)$ is the Laplace transform of $k(.)$, where

$$k(v) \equiv \sum_{\sigma=1}^{\infty} [1-H^\sigma(v)] = \sum_{\sigma=1}^{\infty} \mathbf{P}\{V^\sigma > v\}$$

for $0 < v < \infty$. The function $k(.)$ is positive and monotone decreasing; also

$$\mathbf{P}(V^s > v) = \int_{(v,\infty)} dH^s(V) \leqslant \frac{1}{v} \int_{(v,\infty)} V\, dH^s(V)$$

$$\leqslant \mathbf{E}(V^s)/v = v^{-1} \sum_{\alpha=-\infty}^{\infty} 1/a_\alpha^s,$$

so that $k(v) \leqslant C/v$ for all $v > 0$. If

$$S \equiv \sum\sum 1/a_\alpha^\sigma,$$

then

$$Se^{-\lambda S} < K(\lambda) < S,$$

and so

$$K(0+) = S < \infty \quad \text{and} \quad k(.) \in L(0,\infty).$$

We can now assert that

$$\int_0^t \phi(u)k(t-u)\,du$$

is a bounded continuous function having $\Phi(\lambda)K(\lambda)$ as its Laplace transform, and so $p^{00}(t) \equiv \phi(t)$ *is the unique bounded continuous solution of*

$$\phi(t)+\int_0^t \phi(u)k(t-u)\,du = 1 \quad (t \geqslant 0). \tag{28}$$

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ORDERED ABELIAN GROUPS

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Introduction

KRULL (**7**) considered the relation between valuations and orders of fields. In this paper I apply a generalization of the theory of valuations to the study of ordered abelian groups and thereby derive new proofs of Hahn's embedding and completeness theorems (**5**)† for ordered abelian groups. An account of the generalization is given in (**4**) of which this paper should be regarded as the sequel.

Preliminaries

I give a number of well-known definitions and results concerning ordered abelian groups which lead to a statement of Hahn's theorems. This section is independent of (**4**).

Let Δ be a set. A binary relation '$\leqslant$' of the set Δ is said to be an *order* of Δ (and Δ, more precisely $\{\Delta, \leqslant\}$, is said to be an *ordered* set) if, for each $x, y, z \in \Delta$,

(i) $x \leqslant y$ or $y \leqslant x$ (and hence $x \leqslant x$),

(ii) $x \leqslant y$ and $y \leqslant x$ imply $x = y$,

(iii) $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$.

A subset Σ of an ordered set Δ is said to be *dual-wellordered* if each non-empty subset Σ' of Σ has a maximum element.

By 'group' we shall understand abelian group. The zero element of an arbitrary group is denoted by 0.

Let G be a group. An order '$\leqslant$' of the set G is said to be an *order of the group G* (and G, more precisely $\{G, \leqslant\}$, is said to be an *ordered group*) if, for each $x, y, z \in G$, $x \leqslant y$ implies $x+z \leqslant y+z$. It is easily verified that, if G is an ordered group, then G is locally infinite, i.e. each non-zero element of G is of infinite order.

Let G_1 and G_2 be ordered groups. A mapping f of G_1 into G_2 is said to be an *order-isomorphism* of the ordered group G_1 into the ordered group

† More recent proofs of Hahn's theorems are due to Conrad (**2**) and Hausner and Wendel (**6**) [but see also Clifford (**1**)]. My procedure, although developed independently, resembles Conrad's in so far as he proves a generalization of Hahn's theorems by means of an embedding theorem (for groups valued in his sense).

G_2 if f is an isomorphism of the group G_1 into the group G_2 and, for each $x, y \in G_1$, $x \leqslant y$ (in the order of G_1) if and only if $f(x) \leqslant f(y)$ (in the order of G_2). If f is an isomorphism of G_1 into G_2, it is easy to prove that f is an order-isomorphism if and only if, for each $x \in G_1$, $x > 0$ implies $f(x) > 0$.

A group G is said to be a *rational* group if G is locally infinite and to each element x of G and non-zero integer n there corresponds an element x' of G for which $nx' = x$. It follows that to each element x of G and integers m, n (with n non-zero) there corresponds one and only one element y of G for which $ny = mx$; y is then written $(m/n)x$. A rational group may thus be considered as a linear space over the field of rational numbers.

Let G be a locally infinite group. It is well known that there exists one and, within isomorphism over G, only one rational group G^* which contains G as a subgroup and is such that to each element x of G^* there corresponds a non-zero integer n for which nx is an element of G. G^* is said to be the *rational group* of G.

Let G be an ordered group. Denote by G^* the rational group of the (locally infinite) group G. It is well known that there is one and only one order of the group G^* which coincides on G with the original order of G. G^* so ordered is said to be the *ordered rational* group of G.

Let G be an ordered group. An element x of G is said to be *Archimedean equivalent* to an element y of G if either (i) $|x| \leqslant |y|$, where, for each $z \in G$, $|z| = \max(z, -z)$, and there exists an integer n such that $|nx| \geqslant |y|$ or (ii) $|y| \leqslant |x|$ and there exists an integer n such that $|ny| \geqslant |x|$. Archimedean equivalence is an equivalence relation of the set G. The set of equivalence classes of G is denoted by Δ, more precisely $\Delta(G)$. $\{0\}$, the subset of G consisting of the zero element of G only, is an element of Δ. It is well known that, if G is an Archimedean ordered group (i.e. each pair of non-zero elements of G are Archimedean equivalent), then there is an order-isomorphism of the ordered group G into the ordered additive group of real numbers.

Let G be an ordered group and let Δ be the set of Archimedean equivalence classes of G. Let δ_1 and δ_2 be elements of Δ. Put $\delta_1 \leqslant \delta_2$ if there exists $x_1 \in \delta_1$ and $x_2 \in \delta_2$ such that $|x_1| \leqslant |x_2|$. It is easy to verify that the binary relation '$\leqslant$' is an order of the set Δ. $\{0\}$ is then the minimum element of the ordered set Δ.

Let G_1 and G_2 be ordered groups. If G_1 is an ordered subgroup of G_2, then there is a natural identification of the ordered set $\Delta(G_1)$ with an ordered subset of $\Delta(G_2)$. G_2 is then said to be an *Archimedean extension*

of G_1 if $\Delta(G_1) = \Delta(G_2)$, i.e. to each element x of G_2 there corresponds an element y of G_1 such that x is Archimedean equivalent to y. It is not difficult to show that, if G is an ordered group, then G^*, the ordered rational group of G, is an Archimedean extension of G.

An ordered group G is said to be *Archimedean complete* if there does not exist an ordered group G', properly containing G as an ordered subgroup, which is an Archimedean extension of G. It is clear that, if G is Archimedean complete, then $G = G^*$. It is well known that the ordered additive group of real numbers is Archimedean complete.

Before we state Hahn's theorems we construct, corresponding to an (arbitrary) ordered set Δ with minimum element μ, an ordered group $W(\Delta)$.

Let Δ be an ordered set with minimum element μ. Denote by $W(\Delta)$ the set of all mappings x of Δ into the set of real numbers for which (i) $x(\mu) = 0$ and (ii) Nx, the set of all elements δ of Δ for which $x(\delta) \neq 0$, is a dual-wellordered subset of Δ. Denote by $F(\Delta)$ the set of all elements x of $W(\Delta)$ for which Nx is a finite subset of Δ.

For each $x, y \in W(\Delta)$ and $\delta \in \Delta$, put

$$(x+y)(\delta) = x(\delta)+y(\delta).$$

It is easily verified that, with respect to the binary composition of addition, $W(\Delta)$ is a rational group and that $F(\Delta)$ is a rational subgroup of $W(\Delta)$.

For each $x, y \in W(\Delta)$, put $x \leqslant y$ if $x = y$ or if $x \neq y$ and $x(\delta_0) \leqslant y(\delta_0)$, where $\delta_0 = \max N(x-y)$. It is not difficult to show that the binary relation '$\leqslant$' is an order of the group $W(\Delta)$. Further, the ordered group $W(\Delta)$ is an Archimedean extension of the ordered group $F(\Delta)$, and there are natural identifications of $\Delta(F(\Delta))$ and of $\Delta(W(\Delta))$ with Δ.

Hahn's Embedding Theorem. *Let G be an ordered group. Then there is an order-isomorphism of G into $W(\Delta(G))$.*

Hahn's Completeness Theorem. *An ordered group G is Archimedean complete if and only if there is an order-isomorphism of G onto $W(\Delta(G))$.*

Note. It suffices to prove Hahn's theorems for ordered rational groups (which in the next section are termed 'ordered linear spaces') since $W(\Delta(G))$ is order-isomorphic to $W(\Delta(G^*))$ (since $\Delta(G)$ and $\Delta(G^*)$ are of the same order-type) and G^* is an Archimedean extension of G.

Ordered linear spaces

By 'linear space' we shall understand linear space over the field of rational numbers.

Let L be a linear space. An order '$\leqslant$' of the set L is said to be an *order of the linear space* L (and L, more precisely $\{L, \leqslant\}$, is said to be an ordered linear space) if '$\leqslant$' is an order of the additive group of L. Definitions, concerning ordered groups, of the last section we shall understand to apply to ordered linear spaces.

In this section and the next we consider those relations between orders and valuations of linear spaces necessary for my proofs of Hahn's theorems.

Let L be an ordered linear space and denote by Δ the ordered set of Archimedean equivalence classes of L. Denote by d the mapping of L onto Δ that maps each element x of L onto the element of Δ which contains x. It is easy to verify that d is a valuation of L onto Δ: d is said to be the *natural valuation* of the ordered linear space L. Notice that, if $x, y \in L$ and $d(x-y) < d(x)$ $[= d(y)]$, then $x > 0$ if and only if $y > 0$.

Let δ be an element of Δ. For each $c_1, c_2 \in C[L](\delta)$,† put $c_1 \leqslant c_2$ if there exist $x_1 \in c_1$ and $x_2 \in c_2$ such that $x_1 \leqslant x_2$. It is easy to verify that the binary relation '$\leqslant$' is an Archimedean order of the linear space $C[L](\delta)$.

Let L_1 and L_2 be ordered linear spaces. If L_1 is an ordered linear subspace of L_2, then there is a natural identification of the ordered set $\Delta(L_1)$ with an ordered subset of the ordered set $\Delta(L_2)$. L_2 is then an extension of L_1 (*qua* valued linear spaces). Further, for each $\delta \in \Delta_1 \subseteq \Delta_2$, the Archimedean ordered linear space $C[L_1](\delta)$ is an ordered linear subspace of the Archimedean ordered linear space $C[L_2](\delta)$.

LEMMA 1. *Suppose that*

(i) *L_i and L'_i are ordered linear spaces and L_i is an ordered linear subspace of L'_i for $i = 1, 2$,*

(ii) *L'_i is an immediate extension of L_i, qua valued linear spaces, for $i = 1, 2$,*

(iii) *f is a valuation-isomorphism of L'_1 into L'_2 and f, on L_1, is an order-isomorphism of L_1 onto L_2.*

Then f is an order-isomorphism of L'_1 into L'_2.

Proof. Denote the natural valuation of L'_i by d_i for $i = 1, 2$. Let x' be an element of $L'_1 \setminus L_1$ for which $x' > 0$. It suffices to show that

† $C[L](\delta)$ denotes the difference linear space, corresponding to δ, defined with respect to the valued linear space L.

$f(x') > 0$. There exists an element x of L_1 for which $d_1(x-x') < d_1(x)$ $[= d_1(x')]$ and hence $x > 0$. However $f(x) > 0$ and

$$d_2(f(x)-f(x')) < d_2(f(x)) \quad [= d_2(f(x'))]$$

imply $f(x') > 0$. This completes the proof of the lemma.

Ordered direct sums

Let $\mathscr{L} = \{L(\delta) : \delta \in \Delta\}$ be an ordered system of linear spaces (indexed by an arbitrary ordered set Δ with minimum element μ) and denote the corresponding finite-sum and wellorder-sum by $F(\mathscr{L})$ and $W(\mathscr{L})$ respectively. Suppose that, for each $\delta \in \Delta$, $L(\delta)$ is an Archimedean ordered linear space. Then, putting, for each $x, y \in W(\mathscr{L})$, $x \leqslant y$ if

$$x(d(x-y)) \leqslant y(d(x-y))$$

in the given Archimedean order of $L(d(x-y))$,† it is not difficult to show that

(i) '$\leqslant$' is an order of the linear space $W(\mathscr{L})$;

(ii) the ordered set of Archimedean equivalence classes of the ordered linear space $W(\mathscr{L})$ may be identified with Δ so that the natural valuation of $W(\mathscr{L})$, *qua* ordered linear space, coincides with the valuation d;

(iii) for each $\delta \in \Delta$, the Archimedean ordered linear space $C[W(\mathscr{L})](\delta)$ is an order-isomorph of $L(\delta)$.

If, for each $\delta \in \Delta$ with $\delta \neq \mu$, $L(\delta)$ is the Archimedean ordered linear space of real numbers, then the ordered linear spaces $W(\{L(\delta) : \delta \in \Delta\})$ and $W(\Delta)$ are identical.

LEMMA 2. *Let $\mathscr{L} = \{L(\delta) : \delta \in \Delta\}$ be an ordered system of Archimedean ordered linear spaces. Then there exists an order-isomorphism α of $W(\mathscr{L})$ into $W(\Delta)$ for which $W(\Delta)$ is an Archimedean extension of $\alpha(W(\mathscr{L}))$.*

Proof. To each $\delta \in \Delta$ there corresponds an order-isomorphism, α_δ say, of $L(\delta)$ into the ordered linear space of real numbers. For each $x \in W(\mathscr{L})$ and $\delta \in \Delta$, put $(\alpha(x))(\delta) = \alpha_\delta(x(\delta))$. It is easy to prove that α is an order-isomorphism with the required properties.

Proof of Hahn's theorems

From the note following the statement of Hahn's theorems it suffices to prove the theorems for an ordered linear space L. When L is referred to as a 'valued linear space', it is understood that the reference is to L with its natural valuation.

† d is the valuation of $W(\mathscr{L})$ *qua* wellorder-sum.

Proof of the embedding theorem. Put $\mathscr{L} = \{C[L](\delta): \delta \in \Delta\}$. The valuation-isomorphism ϕ [(4), Lemma 5] of $F(\mathscr{L})$ into L is an order-isomorphism of $F(\mathscr{L})$ into L, *qua* ordered linear spaces since, in the notation of the proof of the lemma, f_δ and g_δ are order-isomorphisms and $\phi(x) > 0$ if and only if $g_{\delta_1}(c_1) > 0$. Denote by ψ the valuation-isomorphism of (4), Theorem 3, with $W(\mathscr{L})$ for M, of L into $W(\mathscr{L})$. It follows from Lemma 1 that ψ is an order-isomorphism of L into $W(\mathscr{L})$, *qua* ordered linear spaces. An application of Lemma 2 completes the proof of the embedding theorem.

The alternative method of proof of (4), Theorem 3 yields an interesting proof of the embedding theorem but I omit the details.

Proof of the completeness theorem. $W(\Delta)$ is an Archimedean extension of $\alpha(W(\mathscr{L}))$ and $W(\mathscr{L})$ is an Archimedean extension of $F(\mathscr{L})$ and, *a fortiori*, of $\psi(L)$. It follows that $W(\Delta)$ is an Archimedean extension of $\alpha\psi(L)$ and it only remains to prove that $W(\Delta)$ is Archimedean complete. To this end let W' be an ordered linear space which is an Archimedean extension of $W(\Delta) = W$, say. Then (i) $\Delta(W') = \Delta(W)$ and (ii) for each $\delta \in \Delta(W) = \Delta(W')$ with $\delta \neq \mu$, $C[W](\delta)$ (which is an order-isomorph of the ordered linear space of real numbers) is an ordered linear subspace of the Archimedean ordered linear space $C[W'](\delta)$, and hence

$$C[W](\delta) = C[W'](\delta)$$

since the ordered linear space of real numbers is Archimedean complete. Thus, with respect to their natural valuations, W' is an immediate extension of W and, since W is pseudo-complete, it follows that $W' = W$ [(4), Lemmas 2 and 3].

Alternatively, we can replace the argument from the words 'To this end . . .' by 'Using the method of (3) it can be directly shown that there exists an Archimedean complete ordered linear space $\overline{W}$ which is an Archimedean extension of $W(\Delta)$.† Defining $\bar{\alpha}$, $\bar{\psi}$ for $\overline{W}$ analogously to α, ψ for L, it is clear that $W(\Delta)$ is an Archimedean extension of $\bar{\alpha}\bar{\psi}(\overline{W})$. Since $\bar{\alpha}\bar{\psi}(\overline{W})$ is Archimedean complete, it follows that $\bar{\alpha}\bar{\psi}(\overline{W}) = W(\Delta)$ and hence that $W(\Delta)$ is Archimedean complete'.

† The method of (3) yields the following result: *If L is an ordered linear space and Δ denotes the corresponding ordered set of Archimedean equivalence classes, then $|L| \leqslant c^{|\Delta|}$ (where c denotes the cardinal number of the set of real numbers).* It follows that there exists an Archimedean extension of L which is Archimedean complete.

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since the ordered linear space of real numbers is Archimedean complete. Thus, with respect to their natural valuations, W' is an immediate extension of W and, since W is pseudo-complete, it follows that $W' = W$ [(**4**), Lemmas 2 and 3].

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† The method of (**3**) yields the following result: *If L is an ordered linear space and Δ denotes the corresponding ordered set of Archimedean equivalence classes, then $|L| \leqslant c^{|\Delta|}$ (where c denotes the cardinal number of the set of real numbers).* It follows that there exists an Archimedean extension of L which is Archimedean complete.

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A NOTE ON THE OSCILLATION OF RIESZ, EULER, AND INGHAM MEANS

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1. Introduction

PENNINGTON (**6**) and I (**9**) have proved, by independent methods,† the following theorem for the Riesz integral means of a function analogous to a theorem of Littlewood's [(**6**) Theorem *t*] for the Cesàro means of a sequence.

THEOREM t_σ. *Suppose that for all $r > 0$ and $x > 0$ the integral*

$$\sigma_r(x) \equiv \frac{A_r(x)}{x^r} = \frac{r}{x^r}\int_0^x (x-u)^{r-1}A(u)\,du$$

exists, $\sigma_0(x) \equiv A_0(x)$ being defined as $A(x)$, which is assumed to be real. Let

$$\limsup_{x\to\infty}\sigma_r(x) = \bar\sigma_r, \qquad \liminf_{x\to\infty}\sigma_r(x) = \underline{\sigma}_r,$$

$$\lim_{r\to\infty}\bar\sigma_r = \bar\sigma_\infty, \qquad \lim_{r\to\infty}\underline{\sigma}_r = \underline{\sigma}_\infty,$$

where the limits in the last line exist since $\bar\sigma_r$ and $\underline{\sigma}_r$ are monotonic functions of r, the first decreasing and the second increasing. Then, provided that

$$\bar\sigma_\infty = \underline{\sigma}_\infty = s \text{ (finite)},$$

we have $\bar\sigma_r = \underline{\sigma}_r = s$ for all sufficiently large r, say $r \geqslant r_0+1$.

Pennington's method is direct and based on an idea used by Ingham (**3**) to simplify the proof of the well-known 'high-indices' theorem for Dirichlet's series due to Hardy and Littlewood; while my method is indirect and employs the Laplace integral in one form. It is my object now to present (i) a direct proof of Theorem t_σ resembling Pennington's in certain respects, but simpler and depending on a classical device of Karamata's [(**2**) § 7.6] in place of Ingham's idea, (ii) a consequence of $\bar\sigma_\infty$ and $\underline{\sigma}_\infty$ being finite but unequal (Theorem 2_σ), following from my method, (iii) a certain correspondence between my indirect proof of Theorem t_σ and that of the analogue of Theorem t_σ for the Euler means

† My method, as originally suggested [(**9**) 377], contained an error which has been pointed out by Pennington [(**6**) 200] and since rectified [(**9**) 254–7; (**9a**)].

Quart. J. Math. Oxford (7), 6 (1956), 64–75.

of a sequence stated below as Theorem t_E [(5)† 204], (iv) an analogue of Theorem t_σ for the Ingham means of type λ recently studied by Pennington (7), stated as Theorem t_I in the concluding section.

THEOREM t_E. *Let $s(n)$ $(n = 0, 1, 2,...)$ be a real sequence for which the sequence of the q-th Euler means $(q \geqslant 0)$ is defined by*

$$E_q(n) = \frac{1}{(q+1)^n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} s(m) \quad (n = 0, 1, 2,...).$$

Let

$$\limsup_{n\to\infty} E_q(n) = \bar{E}_q, \qquad \liminf_{n\to\infty} E_q(n) = \underline{E}_q,$$

so that $\bar{E}_q$, $\underline{E}_q$ are monotonic functions of q, the first decreasing and the second increasing, for which

$$\lim_{q\to\infty} \bar{E}_q = \bar{E}_\infty, \qquad \lim_{q\to\infty} \underline{E}_q = \underline{E}_\infty$$

exist. Then the hypothesis

$$\bar{E}_\infty = \underline{E}_\infty = s \text{ (finite)}$$

ensures that $\bar{E}_q = \underline{E}_q = s$ for all sufficiently large q.

2. Preliminary theorems

Of the two following theorems, the second is a transformation of the first, and the first is the 'positive' Tauberian theorem for functions summable by Riesz integral means of 'infinite order' analogous to the similar theorem for functions summable by certain other methods [(2) Theorem 236].

THEOREM 1. *If $f(u) \geqslant 0$ and integrable over any finite range, and if*

$$\lim_{x\to\infty}(\sup, \inf) \frac{1}{x} \int_0^x f(u)\left(1-\frac{u}{x}\right)^\rho du \sim \frac{S}{\rho+1} \quad (\rho \to \infty),‡ \tag{1}$$

where $x > 0$ and $\rho > 0$, then

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(u)\, du = S. \tag{2}$$

† This reference, which I owe to the referee, is to a paper where Theorem t_E appears with a slight change of notation consisting mainly in the replacement of q by 2^l-1. The proof of Theorem t_E in the paper (5) is based on the particular case of Theorem 1_E stated as part of Theorem 1_E itself and the particular case of Theorem 2_E with $O(1)$ instead of $O_L(1)$ in condition (16). Though these particular cases suffice to prove Theorem t_E, Theorems 1_E and 2_E are relevant here as analogues of Theorems 1_σ and 2_σ, the latter under condition (14) in its second alternative form.

‡ The symbol lim(sup, inf) in any relation is used to assert the truth of the relation when it is replaced by each of lim sup, lim inf in turn.

Proof. (1) gives, when we replace ρ by $\rho(1+r)$ $(r \geqslant 0)$ and then change the variable u to t by the substitution $u/x = 1-e^{-t/\rho}$,

$$\lim_{x\to\infty}(\sup,\inf)\frac{1}{\rho}\int_0^\infty f(x-xe^{-t/\rho})e^{-t(1+r)}e^{-t/\rho}\,dt \sim \frac{S}{\rho(1+r)+1} \quad (\rho\to\infty).$$

Hence, corresponding to any small $\epsilon > 0$, we can find $\rho_0(\epsilon)$ such that, for $\rho(1+r) \geqslant \rho_0(\epsilon)$, and thus *a fortiori*, for $\rho \geqslant \rho_0(\epsilon)$ and any $r \geqslant 0$, we have

$$\left.\begin{aligned}\lim_{x\to\infty}(\sup,\inf)\int_0^\infty f(x-xe^{-t/\rho})e^{-t}e^{-rt}e^{-t/\rho}\,dt &< (S+\epsilon)\int_0^\infty e^{-t}e^{-rt}e^{-t/\rho}\,dt,\\ \text{and } &> (S-\epsilon)\int_0^\infty e^{-t}e^{-rt}e^{-t/\rho}\,dt.\end{aligned}\right. \tag{3}$$

The substitution $u/x = 1-e^{-t/\rho}$ gives us also

$$\frac{1}{x}\int_0^{x(1-e^{-1/\rho})} f(u)\,du = \frac{1}{\rho}\int_0^\infty f(x-xe^{-t/\rho})e^{-t}g(e^{-t})e^{-t/\rho}\,dt, \tag{4}$$

where
$$g(e^{-t}) = \begin{cases} e^t & (0\leqslant t\leqslant 1),\\ 0 & (t>1),\end{cases}$$

so that $g(\tau)$ $(\tau = e^{-t})$ is continuous except at $\tau = e^{-1}$ and bounded for $0 \leqslant \tau \leqslant 1$. Therefore, given any $\delta > 0$, we can use Weierstrass's approximation theorem to find polynomials $p(\tau)$ and $P(\tau)$† such that

$$\left.\begin{aligned} p(\tau) < g(\tau) < P(\tau) \quad (0\leqslant\tau\leqslant 1)\\ \int_0^1 \{P(\tau)-p(\tau)\}\,d\tau = \int_0^\infty e^{-t}\{P(e^{-t})-p(e^{-t})\}\,dt < \delta\\ \text{or, } \textit{a fortiori}, \quad \int_0^\infty e^{-t}\{P(e^{-t})-p(e^{-t})\}e^{-t/\rho}\,dt < \delta\end{aligned}\right\}. \tag{5}$$

From (4) and the first half of (5) we find that

$$\left.\begin{aligned}\frac{1}{x}\int_0^{x(1-e^{-1/\rho})} f(u)\,du &< \frac{1}{\rho}\int_0^\infty f(x-xe^{-t/\rho})e^{-t}P(e^{-t})e^{-t/\rho}\,dt,\\ \text{and } &> \frac{1}{\rho}\int_0^\infty f(x-xe^{-t/\rho})e^{-t}p(e^{-t})e^{-t/\rho}\,dt.\end{aligned}\right. \tag{6}$$

† As, for instance, in Hardy [(2) Theorem 99].

Next we note that, if we multiply the left-hand side of (3) by any real c ($\neq 0$), the right-hand side will be altered to $(Sc \pm \epsilon|c|)\int \dots$. In the first (or second) inequality of (3) altered thus, replace ce^{-rt} successively by each term of $P(e^{-t})$ (or $p(e^{-t})$) and sum the inequalities so obtained. The result is that, for every $\rho \geqslant \rho_0$,

$$\left.\begin{aligned}\limsup_{x\to\infty} \int_0^\infty f(x - xe^{-t/\rho})e^{-t}P(e^{-t})e^{-t/\rho}\,dt \\ < S\int_0^\infty e^{-t}P(e^{-t})e^{-t/\rho}\,dt + \epsilon\int_0^\infty e^{-t}P^*(e^{-t})e^{-t/\rho}\,dt \\ \liminf_{x\to\infty} \int_0^\infty f(x - xe^{-t/\rho})e^{-t}p(e^{-t})e^{-t/\rho}\,dt \\ > S\int_0^\infty e^{-t}p(e^{-t})e^{-t/\rho}\,dt - \epsilon\int_0^\infty e^{-t}p^*(e^{-t})e^{-t/\rho}\,dt\end{aligned}\right\}, \tag{7}$$

where $P^*(e^{-t})$, $p^*(e^{-t})$ are obtained from $P(e^{-t})$, $p(e^{-t})$ respectively by changing every coefficient in the latter two to its modulus; (7) is true *a fortiori* when we replace P^*, p^* by their respective upper bounds M^*, m^* in the interval $0 \leqslant \tau \leqslant 1$. Using (7) with these replacements in the right-hand members of (6), we get, for every $\rho \geqslant \rho_0$,

$$\lim_{x\to\infty}(\sup\inf)\frac{1}{x}\int_0^{x(1-e^{-1/\rho})} f(u)\,du < \frac{S}{\rho}\int_0^\infty e^{-t}P(e^{-t})e^{-t/\rho}\,dt + \frac{\epsilon M^*}{\rho}\int_0^\infty e^{-t}e^{-t/\rho}\,dt,$$

$$\text{and} > \frac{S}{\rho}\int_0^\infty e^{-t}p(e^{-t})e^{-t/\rho}\,dt - \frac{\epsilon m^*}{\rho}\int_0^\infty e^{-t}e^{-t/\rho}\,dt,$$

and therefore, by (5),

$$\lim_{x\to\infty}(\sup,\inf)\frac{1}{x}\int_0^{x(1-e^{-1/\rho})} f(u)\,du < \frac{S}{\rho}\int_0^\infty e^{-t}g(e^{-t})e^{-t/\rho}\,dt + \frac{S\delta}{\rho} + \frac{\epsilon M^*}{\rho},$$

$$\text{and} > \frac{S}{\rho}\int_0^\infty e^{-t}g(e^{-t})e^{-t/\rho}\,dt - \frac{S\delta}{\rho} - \frac{\epsilon m^*}{\rho},$$

i.e. $$\lim_{x\to\infty}(\sup,\inf)\frac{1}{x}\int_0^{x(1-e^{-1/\rho})} f(u)\,du < S(1-e^{-1/\rho}) + (S\delta + \epsilon M^*)\rho^{-1},$$

$$\text{and} > S(1-e^{-1/\rho}) - (S\delta + \epsilon m^*)\rho^{-1}.$$

We now divide each of the above pair of inequalities by $1-e^{-1/\rho}$,

subsequently replacing $x(1-e^{-1/\rho})$ by y and so $x \to \infty$ by $y \to \infty$ in the left-hand member, with the result that, for $\rho \geqslant \rho_0$,

$$\lim_{y\to\infty}(\sup,\inf)\frac{1}{y}\int\limits_0^y f(u)\,du < S+(S\delta+\epsilon M^*)\{\rho(1-e^{-1/\rho})\}^{-1},$$
$$\text{and} > S-(S\delta+\epsilon m^*)\{\rho(1-e^{-1/\rho})\}^{-1}.$$

When we let $\rho \to \infty$ and recall that M^*, m^* are dependent only on δ and independent of ϵ which is arbitrary, this gives

$$S-S\delta \leqslant \lim_{y\to\infty}(\sup,\inf)\frac{1}{y}\int\limits_0^y f(u)\,du \leqslant S+S\delta.$$

Since the polynomials P, p of (5) can be found corresponding to every $\delta > 0$, the above inequalities hold for every $\delta > 0$ and lead at once to our conclusion (2).

THEOREM 2. *If $\alpha \geqslant 0$, $\phi(u) \geqslant 0$ and integrable over any finite range, and if*

$$\lim_{x\to\infty}(\sup,\inf)\frac{1}{x^{\alpha+1}}\int\limits_0^x \phi(u)\Big(1-\frac{u}{x}\Big)^\rho\,du \sim S\,\frac{\Gamma(\alpha+1)\Gamma(\rho+1)}{\Gamma(\alpha+\rho+2)} \quad (\rho\to\infty), \tag{8}$$

where x, $\rho > 0$, then

$$\lim_{x\to\infty}\frac{1}{x}\int\limits_1^x \frac{\phi(u)}{u^\alpha}\,du = S. \tag{9}$$

Remarks on Theorem 2. (i) For the proof which follows, the assumption $\phi(u) \geqslant 0$ for $u \geqslant 1$ is quite sufficient; (ii) with the additional assumption that $\phi(u)/u^\alpha$ is integrable over $(0, 1)$, the lower limit in the integral of (9) may be replaced by 0.

Proof of Theorem 2.† We can write (8) with the lower limit in its integral changed from 0 to 1, since

$$\lim_{x\to\infty}\frac{1}{x^{\alpha+1}}\int\limits_0^1 \phi(u)\Big(1-\frac{u}{x}\Big)^\rho\,du = 0.$$

In (8), rewritten as stated, we can change ρ to $\rho+r$ $(r \geqslant 0)$ and write the factor multiplying S as a beta function. From (8) thus rewritten

† My original proof of Theorem 2 contained a gap which the referee has kindly helped me to fill up.

and changed, it follows that there exists $\rho_0(\epsilon)$, corresponding to any $\epsilon > 0$, such that, for $\rho+r \geqslant \rho_0$ and *a fortiori* for $\rho \geqslant \rho_0$,

$$\left.\begin{aligned}\lim_{x\to\infty}(\sup,\inf)\frac{1}{x^{\alpha+1}}\int\limits_1^\infty \phi(u)\left(1-\frac{u}{x}\right)^{\rho+r}du &< (S+\epsilon)\int\limits_0^1 v^\alpha(1-v)^{\rho+r}\,dv,\\ \text{and} &> (S-\epsilon)\int\limits_0^1 v^\alpha(1-v)^{\rho+r}\,dv.\end{aligned}\right\}\tag{10}$$

Now we have, for $0 < u/x < 1$ and $\alpha > 0$,† the expansion

$$\left(\frac{u}{x}\right)^{-\alpha} = \sum_{r=0}^{N}\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)\Gamma(r+1)}\left(1-\frac{u}{x}\right)^r+\frac{\Gamma(\alpha+N+1)}{\Gamma(\alpha)\Gamma(N+1)}\int\limits_{u/x}^1\left(t-\frac{u}{x}\right)^N t^{-\alpha-N-1}\,dt,$$

where all terms are positive. When we multiply both sides by

$$\phi(u)(1-u/x)^\rho x^{-\alpha-1},$$

then integrate with respect to u from 1 to x, and finally take upper (or lower) limits of both sides as $x\to\infty$, this expansion gives

$$\left.\begin{aligned}&\lim_{x\to\infty}(\sup,\inf)\frac{1}{x}\int\limits_1^x \phi(u)u^{-\alpha}\left(1-\frac{u}{x}\right)^\rho du\\ &\leqslant \sum_{r=0}^{N}\limsup_{x\to\infty}\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)\Gamma(r+1)}x^{-\alpha-1}\int\limits_1^x\phi(u)\left(1-\frac{u}{x}\right)^{\rho+r}du+\limsup_{x\to\infty}R_N,\\ \text{and } &\geqslant \sum_{r=0}^{N}\liminf_{x\to\infty}\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)\Gamma(r+1)}x^{-\alpha-1}\int\limits_1^x\phi(u)\left(1-\frac{u}{x}\right)^{\rho+r}du,\end{aligned}\right\}\tag{11}$$

where

$$0\leqslant R_N = \frac{\Gamma(\alpha+N+1)}{\Gamma(\alpha)\Gamma(N+1)}x^{-\alpha-1}\int\limits_1^x\phi(u)\left(1-\frac{u}{x}\right)^\rho du\int\limits_{u/x}^1\left(t-\frac{u}{x}\right)^N t^{-\alpha-N-1}\,dt$$

$$\leqslant \frac{\Gamma(\alpha+N+1)}{\Gamma(\alpha)\Gamma(N+1)}\frac{1}{x}\int\limits_1^x\frac{d\xi}{\xi^{\alpha+1}}\int\limits_1^\xi\phi(u)\left(1-\frac{u}{\xi}\right)^N du = R'_N \text{ (say)},\tag{12}$$

R'_N resulting from R_N when we replace $(1-u/x)^\rho$ in R_N by 1, then reverse the order of integrations (a process justified by the integrands being non-negative), and lastly put $xt=\xi$. From (12) and the first half of (10) with $\rho+r$ replaced by N, we have at once, for $N\geqslant\rho_0$,

$$\limsup_{x\to\infty}R'_N < \frac{\Gamma(\alpha+N+1)}{\Gamma(\alpha)\Gamma(N+1)}(S+\epsilon)\frac{\Gamma(\alpha+1)\Gamma(N+1)}{\Gamma(\alpha+N+2)} = \frac{(S+\epsilon)\alpha}{\alpha+N+1}.$$

† In the case $\alpha=0$, Theorem 2 reduces to Theorem 1.

Hence, when $N \to \infty$, $\limsup_{x\to\infty} R'_N \to 0$; thus $\limsup_{x\to\infty} R_N \to 0$ necessarily and, using this fact along with (10) and (11), we obtain, for $\rho \geqslant \rho_0$,

$$\lim_{x\to\infty}(\sup,\inf)\frac{1}{x}\int_1^x \phi(u)u^{-\alpha}\left(1-\frac{u}{x}\right)^\rho du$$

$$\leqslant \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)\Gamma(r+1)}(S+\epsilon)\int_0^1 v^\alpha(1-v)^{\rho+r}\,dv,$$

$$\text{and} \geqslant \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)\Gamma(r+1)}(S-\epsilon)\int_0^1 v^\alpha(1-v)^{\rho+r}\,dv.$$

The integrands on the right being non-negative, we can reverse the order of summation and integration in each of the two right-hand members and get, for $\rho \geqslant \rho_0$,

$$\lim_{x\to\infty}(\sup,\inf)\frac{1}{x}\int_1^x \phi(u)u^{-\alpha}\left(1-\frac{u}{x}\right)^\rho du$$

$$\leqslant (S+\epsilon)\int_0^1 (1-v)^\rho\,dv = (S+\epsilon)/(\rho+1),$$

$$\text{and} \geqslant (S-\epsilon)\int_0^1 (1-v)^\rho\,dv = (S-\epsilon)/(\rho+1).$$

Conclusion (9) follows from the above step by an appeal to Theorem 1 with
$$f(u) = \phi(u)u^{-\alpha} \quad (u \geqslant 1), \qquad f(u) = 0 \quad (0 \leqslant u < 1).$$

3. Proof of Theorem t_σ

The hypothesis that σ_∞ is finite ensures the existence of a sufficiently large $r_0 > 0$ and a constant K such that $A_{r_0}(u)+Ku^{r_0} \geqslant 0$ for $u \geqslant 1$ (say).† Now, by a well-known formula,

$$\frac{1}{x^{r_0+1}}\int_0^x \{A_{r_0}(u)+Ku^{r_0}\}\left(1-\frac{u}{x}\right)^\rho du = \frac{\Gamma(r_0+1)\Gamma(\rho+1)}{\Gamma(r_0+\rho+2)}\{\sigma_{r_0+\rho+1}(x)+K\},$$

† The referee points out that a restriction like $u \geqslant 1$ is necessitated by an example such as $A(u) = -u^{-\lambda}$ $(0 < \lambda < 1)$, where $A_{r_0}(u) \sim -cu^{r_0-\lambda}$ as $u \to 0$ (c being a positive constant), so that there is no K such that $A_{r_0}(u)+Ku^{r_0} \geqslant 0$ for $u > 0$ whatever be r_0. The referee also points out that the existence of an r_0 and a corresponding K such that $A_{r_0}(u)+Ku^{r_0} \geqslant 0$ for $u \geqslant 1$ (I) may or may not imply a *least* such r_0. For instance, in the case $A(u) = u^\lambda \sin u$ $(\lambda > 0)$, (I) holds if and only if $r_0 \geqslant \lambda$, while, in the case $A(u) = u^\lambda \sin u \log u$, (I) holds if and only if $r_0 > \lambda$.

i.e.

$$\lim_{x\to\infty}(\sup,\inf)\frac{1}{x^{r_0+1}}\int_0^x \{A_{r_0}(u)+Ku^{r_0}\}\Big(1-\frac{u}{x}\Big)^{\rho}\,du$$
$$\sim \frac{\Gamma(r_0+1)\Gamma(\rho+1)}{\Gamma(r_0+\rho+2)}\{s+K\} \quad (\rho\to\infty),$$

since $\bar{\sigma}_\infty = \underline{\sigma}_\infty = s$. The relation stated above is (8) with

$$\phi(u) = A_{r_0}(u)+Ku^{r_0}, \qquad \alpha = r_0, \qquad S = s+K.$$

Therefore, by Theorem 2,

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x \{\sigma_{r_0}(u)+K\}\,du = s+K$$

if we assume (as we may, without appreciable loss of generality) that $A(u) = O(1)$ as $u \to 0$. The above relation, in the equivalent form $\lim \sigma_{r_0+1}(x) = s$ leads at once to the conclusion of Theorem t_σ.

4. Alternative proof of Theorem t_σ

This alternative proof has two stages represented by the next two theorems. The first of these theorems is of some intrinsic interest though it is needed only in a cruder form (as in **9a**, for instance) in proving Theorem t_σ. The second, obtained by me elsewhere [(**9**) 254, Lemma III] as an extension of a useful result of O. Szász's, is of interest in the present context because it is analogous, under the second alternative of its hypothesis (14), to Theorem 2 with the special choice of $\phi(u)$ employed to prove Theorem t_σ.

THEOREM 1_σ. *In the notation of Theorem t_σ, suppose that $\bar{\sigma}_\infty$, $\underline{\sigma}_\infty$ are finite. Then there is an $r_0 > 0$ such that $\sigma_{r_0}(u)$ is bounded and, whenever $r \geqslant r_0$,*

(i) $\Phi_r(y) \equiv \dfrac{y^{r+1}}{\Gamma(r+1)}\displaystyle\int_0^\infty e^{-yu}A_r(u)\,du$ *converges absolutely for $y > 0$,*

(ii) $\limsup_{y\to+0}\Phi_r(y) = \bar{\sigma}_\infty, \qquad \liminf_{y\to+0}\Phi_r(y) = \underline{\sigma}_\infty.$

In particular, if $\bar{\sigma}_\infty = \underline{\sigma}_\infty = s$, then $\lim_{y\to+0}\Phi_r(y) = s$.

Conclusion (ii) alone need be proved and this is implicit in an argument which I have advanced elsewhere [(**9**) 375–6], following Garten and Knopp [(**1**) proof of Satz 4 b]. Since, for each $r \geqslant r_0$, there is a constant K such that $A_r(u)+Ku^r \geqslant 0$ for $u > 0$ if we assume (with a negligible

loss in generality) that $A(u)$ is bounded over $(0, 1)$, this argument shows that, for $r \geqslant r_0$,

$$\bar{\sigma}_\infty \leqslant \limsup_{y\to+0} \Phi_r(y), \quad \text{while } \bar{\sigma}_\infty \geqslant \limsup_{y\to+0} \Phi_r(y)$$

by a simple argument indicated by Bosanquet (**9a**). Thus we get the first relation in (ii), and similarly the second relation in (ii).

THEOREM 2_σ. *If condition* (i) *of Theorem* 1_σ *holds for* $r = r_0$ *along with*

$$\lim_{y\to+0} \Phi_{r_0}(y) = s, \tag{13}$$

and

$$\textit{either } uA_{r_0}(u)-A_{r_0+1}(u) = O_L(u^{r_0+1}), \quad \textit{or } A_{r_0}(u) = O_L(u^{r_0}), \tag{14}$$

then $$\sigma_{r_0+1}(x) \to s \quad \textit{as } x \to \infty.$$

Theorem 2_σ can be readily deduced from a more general result proved by Jakimovski and myself [(**10**) Theorem II′, Corollary II′.1]. In this method of deduction we first obtain Theorem 2_σ with the first alternative of hypothesis (14) and then prove that (13) and the second alternative of (14) together imply the first alternative of (14). The proof of the last-mentioned fact is subjoined below, not being explicitly given in the paper referred to (**10**).

From the second alternative of (14), there is a constant K such that $A_{r_0}(u)+Ku^{r_0} \geqslant 0$ for $u \geqslant 1$ (say); while, from (13), there are constants C and y_0 $(0 < y_0 < 1)$ such that, for $0 < y \leqslant y_0$,

$$C > \frac{y^{r_0+1}}{\Gamma(r_0+1)} \int_0^1 e^{-yu} A_{r_0}(u)\, du + \frac{y^{r_0+1}}{\Gamma(r_0+1)} \int_1^\infty e^{-yu}\{A_{r_0}(u)+Ku^{r_0}\}\, du$$

$$\geqslant \frac{y^{r_0+1}}{\Gamma(r_0+1)} \int_0^1 e^{-yu} A_{r_0}(u)\, du + \frac{x^{-(r_0+1)}}{\Gamma(r_0+1)} \int_1^x e^{-u/x}\{A_{r_0}(u)+Ku^{r_0}\}\, du$$

$$(x = y^{-1})$$

$$\geqslant \frac{y^{r_0+1}}{\Gamma(r_0+1)} \int_0^1 e^{-yu} A_{r_0}(u)\, du + \frac{e^{-1}x^{-(r_0+1)}}{\Gamma(r_0+1)} \int_1^x \{A_{r_0}(u)+Ku^{r_0}\}\, du.$$

Therefore the second alternative of (14) and (13) together give us

$$\frac{e^{-1}x^{-(r_0+1)}}{\Gamma(r_0+1)} \int_0^x A_{r_0}(u)\, du = O_R(1), \quad \text{i.e.} \quad A_{r_0+1}(x) = O_R(x^{r_0+1}) \quad (x \to \infty),$$

i.e. $$xA_{r_0}(x)-A_{r_0+1}(x) = O_L(x^{r_0+1}) \quad (x \to \infty).$$

5. Proof of Theorem t_E

A proof may be obtained by combining the following two theorems analogous to Theorem 1_σ and Theorem 2_σ with the second alternative of (14).

THEOREM 1_E. *In the notation of Theorem t_E, suppose that $\bar{E}_\infty$, $\underline{E}_\infty$ are finite. Then the Borel transform of $s(n)$, namely,*

$$B[x; s(n)] \equiv e^{-x} \sum_{n=0}^{\infty} s(n)\frac{x^n}{n!} \quad (x > 0)$$

exists and is such that

$$\limsup_{x\to\infty} B[x; s(n)] = \bar{E}_\infty, \qquad \liminf_{x\to\infty} B[x; s(n)] = \underline{E}_\infty.$$

In particular, if $\bar{E}_\infty = \underline{E}_\infty = s$, then $\lim_{x\to\infty} B[x; s(n)] = s$.

Proof. Since our hypothesis implies that $E_q(n)$ is bounded for every sufficiently large q, the theorem becomes in substance one due to Garten and Knopp [(1) Satz 5a] once we establish the existence of $B[x; s(n)]$. To this end we may use the formula

$$s(n) = \frac{1}{(r+1)^n} \sum_{m=0}^{n} \binom{n}{m} r^{n-m} E_q(m), \qquad r = -\frac{q}{q+1},$$

where, since $q+r+qr = 0$, we regard $s(n)$ as the $(q+r+qr)$th transform of itself and the $(q+r+qr)$th transform as the rth transform of the qth transform.† Since $|E_q(n)| \leqslant M$ for a fixed $q \geqslant q_0 > 0$, the above formula shows that

$$|s(n)| \leqslant M\left(\frac{|r|+1}{r+1}\right)^n$$

and that therefore $B[x; s(n)]$ exists as an absolutely convergent series for $x > 0$.

THEOREM 2_E. *If, in the notation of Theorem 1_E, $B[x; s(n)]$ exists for $x > 0$, and*

$$B[x; s(n)] \to s \quad \text{as } x \to \infty, \tag{15}$$

and if there is a $q_0 > 0$ such that, in the notation of Theorem t_E,

$$E_{q_0}(n) = O_L(1) \quad \text{as } n \to \infty, \tag{16}$$

then, for every $\delta > 0$, $\qquad E_{q_0+\delta}(n) \to s.$

Proof. A theorem proved by Garten and Knopp [(1) Satz 4a] asserts in effect that, by virtue of (15) and (16), we have, for every (fixed) $k > 0$,

$$E_k[E_{q_0}(n)] \equiv E_q(n) = O(1), \qquad q = q_0+k+q_0 k. \tag{17}$$

† Hardy's proof of this result [(2) Theorem 119] is applicable to any real q and r different from -1.

Since it is well known [(**2**) proof of Theorem 128] that

$$B[(q+1)x; E_q(n)] = B[x; s(n)],$$

(15) can be written: $$B[(q+1); E_q(n)] \to s. \tag{18}$$

By one case of a theorem of Meyer–König's† [(**4**) Satz 25], first noticed by Pitt [(**8**) 520], it follows from (17) and (18) that, for every $l > 0$,

$$E_l[E_q(n)] \equiv E_r(n) \to s,$$

where $$r = q+l+ql = q_0+(q_0+1)(k+l)+(q_0+1)kl.$$

After this the desired conclusion is obvious since, whatever be $\delta > 0$, we can choose k, l so that $r = q_0+\delta$; e.g. we can choose

$$k = l = [\delta(q_0+1)^{-1}+1]^{\frac{1}{2}}-1.$$

6. On the oscillation of Ingham means of type λ

Let $\sum\limits_{1}^{\infty} a_n$ be a real series and λ denote the sequence:

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots \quad (\lambda_n \to \infty).$$

If, for $\kappa \geqslant 0$ and $u > 0$,

$$\chi_\kappa(u) = (\kappa+1)u \sum_{n<1/u} (1-nu)^\kappa,$$

then the Ingham mean of $\sum a_n$, of order κ and type λ, has been defined [(**7**) § 2] as

$$S_\kappa(x) \equiv \frac{(\kappa+1)T_\kappa(x)}{x^{\kappa+1}} = \begin{cases} 0 & (x = 0), \\ \sum\limits_{\lambda_n<x} a_n\, \chi_\kappa(\lambda_n/x) & (x > 0); \end{cases}$$

and Ingham summability of order κ and type λ for the series $\sum a_n$ has been defined as the convergence of $S_\kappa(x)$ as $x \to \infty$.

The following theorem for Ingham means is analogous to Theorem t_σ.

THEOREM t_I. *For $S_\kappa(x)$ defined as above, let*

$$\limsup_{x\to\infty} S_\kappa(x) = \overline{S}_\kappa, \qquad \liminf_{x\to\infty} S_\kappa(x) = \underline{S}_\kappa,$$

$$\lim_{\kappa\to\infty} \overline{S}_\kappa = \overline{S}_\infty, \qquad \lim_{\kappa\to\infty} \underline{S}_\kappa = \underline{S}_\infty,$$

the limits in the last line existing since $\overline{S}_\kappa$ is a monotonic decreasing function of κ and $\underline{S}_\kappa$ a monotonic increasing function [(**7**) Theorem 1]. *Then, provided that*

$$\overline{S}_\infty = \underline{S}_\infty = s \ (\textit{finite}),$$

we have $\overline{S}_\kappa = \underline{S}_\kappa = s$ for all sufficiently large κ.

† The case in question of Meyer-König's theorem is that, for bounded $s(n)$, (15) implies $E_l(n) \to s$, $l > 0$.

This theorem may be deduced from Theorems 1, 2, exactly as its analogue is deduced in § 3, by using the fact [(7) Lemma 1] that the sums $(\kappa+1)T_\kappa(x)$ are subject to the same law of iteration as $A_{\kappa+1}(x)$.

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ON THE LATENT VECTORS AND CHARACTERISTIC VALUES OF PRODUCTS OF PAIRS OF SYMMETRIC IDEMPOTENTS

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FOR a symmetric idempotent matrix $\mathbf{e}$, $\mathbf{e}' = \mathbf{e}$, $\mathbf{e}^2 = \mathbf{e}$. Let $\mathbf{e}$, $\mathbf{f}$ be a pair of symmetric idempotents with real elements. Then the products $\mathbf{ef}$, $\mathbf{fe}$ are transposes of each other: $(\mathbf{ef})' = \mathbf{f}'\mathbf{e}' = \mathbf{fe}$; so they have the same characteristic values, with the same multiplicities; and these values and multiplicities characterize a symmetrical relation between $\mathbf{e}$ and $\mathbf{f}$.

THEOREM I. *The characteristic values of* $\mathbf{ef}$ *are all real, at least zero and at most unity.*

The characteristic values of $\mathbf{ef} = \mathbf{eeff}$ are the same as those of $\mathbf{feef} = (\mathbf{ef})'\mathbf{ef}$, which is symmetric and non-negative definite; hence they are real and non-negative; and they are all zero if and only if $\mathbf{ef} = \mathbf{0}$.

If $\mathbf{ef} \neq \mathbf{0}$, let λ be any non-zero characteristic value, necessarily real, and let $\mathbf{U}$ be a real latent vector which belongs to it, so that $\mathbf{efU} = \mathbf{U}\lambda$. Since $\lambda \neq 0$,
$$\mathbf{eU} = \mathbf{eefU}\lambda^{-1} = \mathbf{efU}\lambda^{-1} = \mathbf{U}.$$
Hence
$$\mathbf{U}'\mathbf{fU} = \mathbf{U}'\mathbf{efU} = \mathbf{U}'\mathbf{U}\lambda,$$
which gives $\lambda = \mathbf{U}'\mathbf{fU}/\mathbf{U}'\mathbf{U}$. But the stationary values of $\mathbf{X}'\mathbf{fX}/\mathbf{X}'\mathbf{X}$ are the characteristic values of $\mathbf{f}$, as appears by methods of the calculus; and these values are all zero or unity. Since $\mathbf{X}'\mathbf{fX}$ is a continuous function in the closed region $\mathbf{X}'\mathbf{X} = 1$, the maximum and minimum values are among these stationary values, and so it follows that
$$0 \leqslant \lambda \leqslant 1.\dagger$$

For any pair of vectors $\mathbf{U}$, $\mathbf{V}$,
$$\cos^2(\mathbf{U}, \mathbf{V}) = (\mathbf{U}'\mathbf{V})^2/(\mathbf{U}'\mathbf{U})(\mathbf{V}'\mathbf{V}).$$

THEOREM II. *If* $\mathbf{U}_\lambda$ *is a latent vector of* $\mathbf{ef}$ *for a non-zero characteristic value* $\lambda = \alpha, \beta$ *and if* $\mathbf{V}_\lambda = \mathbf{fU}_\lambda$, *then*

(i) $\mathbf{V}_\lambda$ *is a latent vector of* $\mathbf{fe}$ *for the characteristic value* λ;

(ii) $\lambda = \cos^2(\mathbf{U}_\lambda, \mathbf{V}_\lambda)$;

† With reference to Afriat (**1**), another proof is as follows:
$$|\mathbf{e}|^* = |\mathbf{e}'\mathbf{e}|^* = |\mathbf{e}|^{*2},$$
so that $|\mathbf{e}|^* = 1$. Hence $|\lambda| \leqslant |\mathbf{ef}|^* \leqslant |\mathbf{e}|^*|\mathbf{f}|^* = 1$.

Quart. J. Math. Oxford (2), 7 (1956), 76-78.

(iii) *if* $\mathbf{U}'_\alpha \mathbf{U}_\beta = 0$, *then* $\mathbf{U}'_\alpha \mathbf{V}_\beta = 0$, $\mathbf{U}'_\beta \mathbf{V}_\alpha = 0$, $\mathbf{V}'_\alpha \mathbf{V}_\beta = 0$;
(iv) *if* $\alpha \neq \beta$, *then* $\mathbf{U}'_\alpha \mathbf{U}_\beta = 0$.

Thus, $\mathbf{efU}_\lambda = \mathbf{U}_\lambda \lambda$, for $\lambda = \alpha, \beta \neq 0$. Hence

$$\mathbf{feV}_\lambda = \mathbf{fefU}_\lambda = \mathbf{fU}_\lambda \lambda = \mathbf{V}_\lambda \lambda,$$

which proves (i). Now

$$\mathbf{eU}_\lambda \lambda = \mathbf{eefU}_\lambda = \mathbf{efU}_\lambda = \mathbf{U}_\lambda \lambda,$$

so that $\mathbf{eU}_\lambda = \mathbf{U}_\lambda$ since $\lambda \neq 0$. Therefore

$$\mathbf{U}'_\lambda \mathbf{U}_\mu \mu = \mathbf{U}'_\lambda \mathbf{efU}_\mu = \mathbf{U}'_\lambda \mathbf{fU}_\mu;$$

and, since $$\mathbf{U}'_\lambda \mathbf{V}_\mu = \mathbf{U}'_\lambda \mathbf{fU}_\mu = \mathbf{U}'_\lambda \mathbf{f}'\mathbf{fU}_\mu = \mathbf{V}'_\lambda \mathbf{V}_\mu,$$

it follows that $$\mathbf{U}'_\lambda \mathbf{U}_\mu \mu = \mathbf{U}'_\lambda \mathbf{V}_\mu = \mathbf{V}'_\lambda \mathbf{V}_\mu,$$

for $\lambda, \mu = \alpha, \beta$. Now (iii) follows immediately. By interchanging λ and μ, we have $\mathbf{U}'_\alpha \mathbf{U}_\beta \alpha = \mathbf{U}'_\alpha \mathbf{U}_\beta \beta$, so that $\mathbf{U}'_\alpha \mathbf{U}_\beta = 0$ if $\alpha \neq \beta$, which proves (iv). Finally,

$$\cos^2(\mathbf{U}_\lambda, \mathbf{V}_\lambda) = (\mathbf{U}'_\lambda \mathbf{V}_\lambda)^2/(\mathbf{U}'_\lambda \mathbf{U}_\lambda)(\mathbf{V}'_\lambda \mathbf{V}_\lambda) = \lambda,$$

which proves (ii).

THEOREM III. *The nullity of* $\mathbf{ef}-\lambda\mathbf{1}$ *is equal to the multiplicity of* λ *as a characteristic value of* $\mathbf{ef}$.

Since the characteristic equation of a product of matrices depends only on the cyclical order of the factors, the characteristic values λ of $\mathbf{ef} = \mathbf{e(eff)}$ are the same as the characteristic values of $\mathbf{(eff)e} = \mathbf{(fe)'fe}$, with the same multiplicities m_λ. Also, provided that $\lambda \neq 0$, in which case $\mathbf{eU} = \mathbf{U}$, it appears that $\mathbf{efU} = \mathbf{U}\lambda$ is equivalent to $\mathbf{efeU} = \mathbf{U}\lambda$, so that the null spaces of $\mathbf{ef}-\lambda\mathbf{1}$ and $\mathbf{efe}-\lambda\mathbf{1}$ are identical. But, by the symmetry of $\mathbf{efe}$, the nullity of the latter is m_λ. Hence the theorem is proved when $\lambda \neq 0$. It remains now to observe that $\mathbf{ef}$ and $\mathbf{efe} = \mathbf{ef(ef)'}$ have the same nullity, since $\mathbf{X'ef} \neq \mathbf{0}$ is equivalent, $\mathbf{X'ef(ef)'X} = \mathbf{0}$, and therefore to $\mathbf{X'ef(ef)'} = \mathbf{0}$, and the nullity of the latter is m_0.

THEOREM IV. *If* $\mathscr{E}, \mathscr{F}$ *are the ranges of* $\mathbf{e}$, $\mathbf{f}$ *then*

$$\mathscr{E} \ominus \mathbf{e}\mathscr{F} = \mathscr{E} \ominus \mathscr{F} = \mathscr{E} \ominus \mathbf{f}\mathscr{E}.$$

If $\mathbf{U} = \mathbf{eV}$, then $\mathbf{eU} = \mathbf{U}$. Hence $\mathbf{U} = \mathbf{eV}$ and $\mathbf{feU} = \mathbf{0}$ is equivalent to $\mathbf{U} = \mathbf{eV}$ and $\mathbf{fU} = \mathbf{0}$. This gives $\mathscr{E} \ominus \mathbf{e}\mathscr{F} = \mathscr{E} \ominus \mathscr{F}$. Also, if $\mathbf{eU} = \mathbf{U}$, then $\mathbf{fU} = \mathbf{0}$ is equivalent to $\mathbf{feU} = \mathbf{0}$; and $\mathbf{0} = \mathbf{efU} = \mathbf{(fe)'feU}$ implies $\mathbf{feU} = \mathbf{0}$. Hence $\mathbf{U} = \mathbf{eV}$ and $\mathbf{efU} = \mathbf{0}$ is equivalent to $\mathbf{U} = \mathbf{eV}$ and $\mathbf{fU} = \mathbf{0}$, and this gives $\mathscr{E} \ominus \mathscr{F} = \mathscr{E} \ominus \mathbf{f}\mathscr{E}$.

THEOREM V. *If $\mathscr{E},\mathscr{F}$ are the ranges of* $\mathbf{e}$, $\mathbf{f}$ *and* $\lambda_1,\dots,\lambda_r$ *are all the non-zero characteristic values of* $\mathbf{ef}$, *with repetitions according to their multiplicities, then there exist unit vectors* $\mathbf{E}_1,\dots,\mathbf{E}_r$ *and* $\mathbf{F}_1,\dots,\mathbf{F}_r$ *which span* $\mathbf{e}\mathscr{F}$ *and* $\mathbf{f}\mathscr{E}$ *respectively, and are such that*

$$\mathbf{E}_i'\mathbf{E}_j = \mathbf{0}, \qquad \mathbf{E}_i'\mathbf{F}_j = \mathbf{0}, \qquad \mathbf{F}_i'\mathbf{F}_j = \mathbf{0} \quad (i \neq j); \qquad \mathbf{E}_i'\mathbf{F}_i = \rho_i,$$

where $\rho_i^2 = \lambda_i$.

The spaces $\mathbf{e}\mathscr{F}$ and $\mathbf{f}\mathscr{E}$ are of dimension equal to the rank of $\mathbf{ef}$ which, by Theorem III, is equal to the total multiplicity r of the non-zero characteristic values of $\mathbf{ef}$. It follows from Theorems II (iv) and III that there exist orthogonal unit vectors $\mathbf{E}_1,\dots,\mathbf{E}_r$ such that $\mathbf{efE}_i = \mathbf{E}_i\lambda_i$. Each of these lies in $\mathbf{e}\mathscr{F}$ since each $\lambda_i \neq 0$; and their range has the same dimension, and hence is identical with $\mathbf{e}\mathscr{F}$. Let $\mathbf{F}_i^* = \mathbf{fE}_i$; and let $\mathbf{F}_i$ be the unit vector obtained by normalizing $\mathbf{F}_i^*$. Then it follows from Theorem II (iii) that $\mathbf{F}_1,\dots,\mathbf{F}_r$ are orthogonal, and their range, in $\mathbf{f}\mathscr{E}$, has the same dimension, and hence is identical with $\mathbf{f}\mathscr{E}$. The final conclusion is now obtained from Theorem II, (ii) and (iii).

THEOREM VI. *There exist matrices* $\mathbf{E}$, $\mathbf{F}$ *such that*

$$\mathbf{e} = \mathbf{EE}', \quad \mathbf{f} = \mathbf{FF}', \quad \mathbf{E}'\mathbf{E} = 1, \quad \mathbf{F}'\mathbf{F} = 1, \quad \mathbf{E}'\mathbf{F} = \begin{pmatrix} \rho_1 \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_r & \\ & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Let $\mathbf{E}_1,\dots,\mathbf{E}_r$ and $\mathbf{F}_1,\dots,\mathbf{F}_r$ be as determined in Theorem V and take orthogonal unit vectors $\mathbf{E}_{r+1},\dots,\mathbf{E}_p$ and $\mathbf{F}_{r+1},\dots,\mathbf{F}_q$ spanning $\mathscr{E}\ominus\mathbf{e}\mathscr{F}$ and $\mathscr{F}\ominus\mathbf{f}\mathscr{E}$. Let $\mathbf{E} = (\mathbf{E}_1\dots\mathbf{E}_p)$, $\mathbf{F} = (\mathbf{F}_1\dots\mathbf{F}_q)$. Then, by Theorem IV, $\mathbf{E}_i'\mathbf{F}_j = \mathbf{0}$ for $i = r+1,\dots,p$ and $j = 1,\dots,q$ and again for $i = 1,\dots,p$ and $j = r+1,\dots,q$; and now, by Theorem V,

$$\mathbf{E}'\mathbf{E} = 1, \ \mathbf{F}'\mathbf{F} = 1, \ \mathbf{E}'\mathbf{F} = \begin{pmatrix} \rho_1 \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho_r & \\ & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Finally, $\mathbf{EE}'$, $\mathbf{FF}'$ are symmetric idempotents with the same range as $\mathbf{e}$, $\mathbf{f}$; hence they are identified with $\mathbf{e}$, $\mathbf{f}$.†

† These results have application, and had their original suggestion, in the canonical correlation theory of Hotelling (**2**).

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TWO EXPANSIONS OF THE BESSEL FUNCTION $K_n(z)$ IN TERMS OF $I_n(z)$

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IN a recent issue of this journal C. P. Singer (**3**) has given an interesting expansion for the Bessel function $K_0(z)$ in terms of $I_{2n}(z)$ as well as a generalization for $K_n(z)$. His article has suggested further research in the course of which it has been noted that two different expansions of $K_n(z)$ in terms of $I_n(z)$ may be obtained, the first directly from Neumann's expansion (**1**) of $Y^{(n)}(z)$ and the second from expansions given by Schläfi (**2**). Both reduce to the expression given by Singer when $n = 0$. In the first expansion of $K_n(z)$ to be noted here the $I_n(z)$ appear linearly rather than in the denominators, as in (**3**), which may be of advantage for analytical work. In the second, the poles of $K_n(z)$ are made manifest as a finite sum of terms of the form z^{-n}, the remaining infinite sum again being linear in the $I_n(z)$ and having no singularities.

Explicitly, starting from the relations between the various Bessel functions [(**4**) 73 § 3.6 (1); 71 (8); 78 (8)]

$$H_\nu^{(1)}(z) = J_\nu(z)+iY_\nu(z), \tag{1}$$

$$Y^{(n)}(z) = \tfrac{1}{2}\pi Y_n(z)+(\log 2-\gamma)J_n(z), \tag{2}$$

$$K_\nu(z) = \tfrac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i}H_\nu^{(1)}(iz), \tag{3}$$

we have
$$K_n(z) = (-1)^n(\tfrac{1}{2}\pi i+\log 2-\gamma)I_n(z)-i^nY^{(n)}(iz) \tag{4}$$

and
$$K_n(z) = (-1)^n\,\tfrac{1}{2}\pi i I_n(z)-\tfrac{1}{2}\pi i^n Y_n(iz). \tag{5}$$

Then, substituting Neumann's expansion [(**4**) 71 (1)] of $Y^{(n)}(z)$ in (4) we have

$$K_n(z) = (-1)^n[\psi(n+1)-\log\tfrac{1}{2}z]I_n(z)+\tfrac{1}{2}\sum_{n=0}^{n-1}\frac{(-1)^m n!}{(n-m)m!}(\tfrac{1}{2}z)^{m-n}I_m(z)+$$
$$+(-1)^n\sum_{m=1}^{\infty}\left[\frac{1}{m}+\frac{1}{n+m}\right]I_{n+2m}(z), \tag{6}$$

in which the finite sum is to be omitted when $n = 0$; in this case we obtain $K_0(z)$ as given in (**3**). Here all the singular terms are confined to the logarithm and the finite sum.

On the other hand Schläfi has noted [(**4**) 340 (1)] that $Y_n(z)$ can be

expressed as the sum of four functions, each of which has fairly simple properties,

$$\pi Y_n(z) = 2[\log \tfrac{1}{2}z - \psi(1)]J_n(z) - S_n(z) + T_n(z) - 2U_n(z), \tag{7}$$

where [(**4**) 258 (1), (2); 343 (1); 71 (3)]

$$S_n(z) = \sum_{m=0}^{<\frac{1}{2}n} \frac{(n-m-1)!}{m!} (\tfrac{1}{2}z)^{-n+2m}, \tag{8}$$

$$T_n(z) = \sum_{m=1}^{\infty} \frac{1}{m} [J_{n+2m}(z) - J_{n-2m}(z)], \tag{9}$$

$$U_n(z) = [\psi(n+1) - \psi(1)]J_n(z) + \sum_{m=1}^{\infty} (-1)^m \left[\frac{1}{m} + \frac{1}{n+m}\right] J_{n+2m}(z). \tag{10}$$

Substituting (7)–(10) in (5) we have

$$K_n(z) = (-1)^n[\psi(n+1) - \log \tfrac{1}{2}z]I_n(z) +$$
$$+\tfrac{1}{2} \sum_{m=0}^{<\frac{1}{2}n} \frac{(-1)^m(n-m-1)!}{m!} (\tfrac{1}{2}z)^{-n+2m} + (-1)^n \sum_{m=1}^{\infty} \left[\frac{1}{m} + \frac{1}{n+m}\right] I_{n+2m}(z) -$$
$$-\tfrac{1}{2}(-1)^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} [I_{n+2m}(z) - I_{n-2m}(z)], \tag{11}$$

where the finite sum is to be omitted when $n = 0$, in which case we again obtain $K_0(z)$ as given in (**3**). As in (6), all the singular terms are confined to the logarithm and the finite sum. Here, however, the coefficients of the poles of various orders are put in evidence.

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